THE SOLUTION OF SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS

by

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Introduction

Engineering problems involving a continuum, such as the elastic behaviour of solids, the motion of fluids and the movement and transport of heat and contaminants, are most often modelled with partial differential equations. In actual fact, all of these models are an approximation to reality, but the correspondence between solutions of these equations and reality is often so close that these models can be used both to study the physical behaviour of reality and to also predict future behaviour for engineering design purposes. Therefore, even students who may never actually calculate a new and original solution to a set of equations will find exposure to this topic helpful for two reasons: first, it provides an efficient way to study and understand the physical behaviour of engineering problems, and, second, analytical and numerical solutions of partial differential equations provide the basis for a large number of computer software programs that are used for engineering design.

Like any of the mathematics that you have studied during your academic career, the key to mastering material given in these lectures lies in understanding rather than memorization. You will need to understand both why and how some of these techniques work, and then you will need to reinforce this understanding by immediately applying the techniques to examples. These examples will be provided for you by assigned homework problems, so that homework problems are an important part of your learning process. Thus, any attempt to either avoid or take shortcuts in homework assignments will ultimately make it much more difficult, if not impossible, for you to learn the material at a later date.

The material covered in these lectures is usually considered to be at the more advanced end of the mathematics spectrum that engineering students are exposed to at the undergraduate level. Consequently, this material will require you to understand and use many of the skills that you have learned in previous mathematics courses. For example, you will find it very difficult to do well in this section of the course unless you have already mastered algebra, ordinary and partial differentiation, elementary integration and solution techniques for relatively simple ordinary differential equations. We will also make use of material that you have just studied in this course, including vector field theory and Fourier series.

Lecture 1

A Partial Differential Equation of Engineering Interest

An application area that has become increasingly important to New Zealand civil engineers in the last 25 years or so is the study of groundwater resource engineering. Most groundwater flow problems are solved by using the following form of Darcy's law for isotropic aquifers:[1]

$$\overline{V} = -K\overline{\nabla}h \tag{1.1}$$

where $\overline{V} = u\hat{i} + v\hat{j} + w\hat{k}$ =flux velocity or specific discharge (flow divided by the sum of the area of pores plus the area of solids in the cross section), K = coefficient of permeability or hydraulic conductivity, h = piezometric head and $\overline{\nabla}h = \hat{i}\partial h/\partial x + \hat{j}\partial h/\partial y + \hat{k}\partial h/\partial z$. We will take z to be positive in the upward direction, so that the (x, y) plane is horizontal. Therefore, $h = p/\rho g + z$, where p = pressure, $\rho =$ water mass density (= 1,000 kg/m^3) and g = acceleration of gravity (= 9.81 m/s^2).

Most groundwater flow solutions in water resource engineering make the Dupuit approximation, which assumes that the velocity component in the vertical direction, w, is small relative to the horizontal velocity components, u and v, and can be neglected. This has several important consequences. First, the z component of Eq.(1.1) shows that

$$w = -\,K\frac{\partial h}{\partial z} = 0 \tag{1.2}$$

which shows that h does not change with z within an aquifer. The physical interpretation of this is that the free surface level (piezometric head, h) in a well will not change as the well is deepened, and it also means that h(x, y, t) is given by the z coordinate of the free surface for an unconfined aquifer. Second, since $\overline{V} = u\hat{i} + v\hat{j} = -K\left(\hat{i}\partial h/\partial x + \hat{j}\partial h/\partial y\right)$, and since h does not change with z, the horizontal velocity components, u and v, do not change with z. This

[1] Darcy's law has been shown experimentally to hold for Reynolds numbers less than about one to ten, where the Reynolds number is given by VD/ν (V = flux velocity magnitude, D = mean grain diameter of the porous matrix and $\nu =$ kinematic viscosity of water). This seems to include most problems of interest to civil engineers. A plausible explanation for the theoretical origin of Darcy's law is given by Eqs.(7.27)-(7.32) in Hunt (1995).

means that we can obtain a vector giving flow per unit arc length in the (x, y) plane by multiplying Eq.(1.1) by the vertical saturated thickness, B, of the aquifer.

$$\overline{q}\left(\equiv \hat{i}q_x + \hat{j}q_y\right) = -T\overline{\nabla}h \tag{1.3}$$

where $\overline{q} = B\overline{V}$ = flow vector per unit arc length in the (x, y) plane, T = KB =transmissivity of the aquifer (units of m^2/s) and $\overline{\nabla}h = \hat{i}\partial h/\partial x + \hat{j}\partial h/\partial y$ has components only in the horizontal direction since h depends, in general, upon x, y and time, t, but not z. Eq.(1.3), which is the form of Darcy's law that is used in most groundwater resource problems, states that the discharge vector, $\overline{q}$, is perpendicular to contours of constant h and is in the direction of decreasing h.

Eq.(1.3) must be combined with a continuity equation to obtain one equation with h(x,y,t) as its only unknown. If we consider a small control volume, shown in Fig. 1.1, bounded laterally by a closed vertical surface, Γ, on the bottom by an impermeable horizontal plane and on top by either a free surface or a second impermeable plane, then a control volume form of the continuity equation is given by

$$\oint_{\Gamma} \overline{q}\cdot\left(-\hat{n}\right) ds + \iint_{A} R\ dx\ dy = \frac{\partial}{\partial t}\iint_{A} \sigma\ B\ dx\ dy \tag{1.4}$$

where $\hat{n} =$ outward normal vector to Γ (so that $-\hat{n} =$ inward normal to Γ), R = vertical flux velocity for recharge from rainfall or irrigation water introduced through the free surface, $\sigma =$ porosity, B = aquifer saturated thickness and A = area in the (x, y) plane bounded by the vertical surface Γ. The left side of Eq.(1.4) gives the net flow of water through the boundary surfaces of the control volume, and the right side is the rate at which water is stored within the control volume.

When the top boundary is a free surface, then σ is the "effective" porosity of the aquifer, which we will assume constant. (The effective porosity is smaller than the actual porosity since some of the pore spaces are unconnected and since surface tension effects cause some water to be left behind as a free surface drops.) Thus, σ typically has an order of magnitude of about 0.1 for an unconfined aquifer. Also, since h(x, y, t) gives the free surface elevation in an unconfined aquifer, and since Fig. 1.1 shows that $z_0(x,y)$ = elevation of the bottom aquifer boundary, we have $B(x,y,t) = h(x,y,t) - z_0(x,y)$ and $\partial h/\partial t = \partial B/\partial t$. Thus, the right side of Eq.(1.4) can be rewritten for an unconfined aquifer in the form

$$\frac{\partial}{\partial t}\iint_{A} \sigma\ B\ dx\ dy = \iint_{A} \sigma \frac{\partial h}{\partial t}\ dx\ dy \tag{1.5}$$

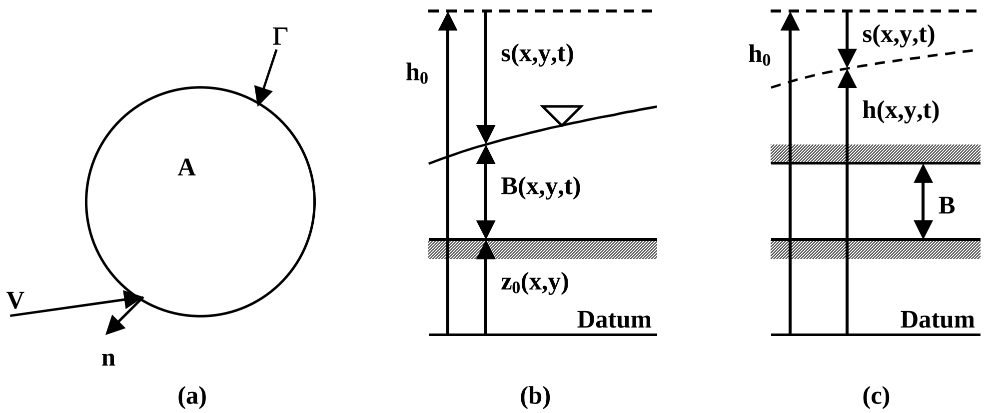

Fig. 1.1. A control volume shown (a) in plan view and in elevation view for (b) an unconfined aquifer and (c) a confined aquifer.

On the other hand, a confined aquifer has no free surface. In this case storage can be increased within the control volume only by an elastic expansion of pore space. In this case, both σ and B will change only if h changes, and the chain rule for differentiation allows us to rewrite the right side of Eq.(1.4) as follows:

$$\frac{\partial}{\partial t}\iint_A \sigma\ B\ dx\ dy = \iint_A \frac{d(\sigma B)}{dh}\frac{\partial h}{\partial t}\ dx\ dy = \iint_A S\frac{\partial h}{\partial t}\ dxdy \qquad (1.6)$$

where $S \equiv d(\sigma B)/dh$ denotes a dimensionless elasticity constant that typically has an order of magnitude of 10^{-4} to 10^{-5}.

Using either Eq.(1.5) or Eq.(1.6) in the right side of Eq.(1.4) and using the divergence theorem to rewrite the first integral on the left side as an area integral over A allows Eq.(1.4) to be rewritten in the following form:

$$\iint_A \left[\overline{\nabla}\cdot\overline{q} + S\frac{\partial h}{\partial t} - R\right] dx\ dy = 0 \qquad (1.7)$$

where S denotes either an effective porosity for an unconfined aquifer or an elasticity constant for a confined aquifer. Eq.(1.7) holds for all arbitrary choices for A, which means that the integrand must vanish at all points within A. Thus, the differential equation form of the continuity equation is

$$\overline{\nabla} \cdot \overline{q} + S\frac{\partial h}{\partial t} - R = 0 \tag{1.8}$$

Substituting the right side of Eq.(1.3) for $\overline{q}$ in Eq.(1.8) gives a fairly general form of the differential equation for h.

$$\overline{\nabla} \cdot \left(T\overline{\nabla}h\right) = S\frac{\partial h}{\partial t} - R \tag{1.9}$$

We will specialize this equation by making the further assumption that T is constant, which means that the aquifer is assumed to be homogeneous for the horizontal scales that we will consider in these lectures. Since T = KB for an unconfined aquifer, this also means that the equation for unconfined flow has been linearized by neglecting changes in B and T caused by calculated changes in h. This will be a valid approximation if calculated changes in h for an unconfined aquifer are small compared with the saturated aquifer thickness, B. In this case, Eq.(1.9) reduces to the following linear equation:

$$T\nabla^2 h = S\frac{\partial h}{\partial t} - R \tag{1.10}$$

Finally, groundwater hydrologists often put Eq.(1.10) in a slightly different form for many calculations by setting

$$h = h_0 - s \tag{1.11}$$

where $h_0 =$ constant value of h that exists prior to the time of interest (i.e. $h = h_0$ for $t \le 0$) and s = piezometric drawdown, as shown in Fig. 1.1. Thus, s is positive in the downward direction, and Eq.(1.10) takes the following form:

$$T\nabla^2 s = S\frac{\partial s}{\partial t} + R \tag{1.12}$$

where $\nabla^2 s = \partial^2 s / \partial x^2 + \partial^2 s / \partial y^2$.

It has been pointed out that S in Eq.(1.12) denotes either an effective porosity, with a magnitude of about 0.1, for an unconfined aquifer or an elasticity constant, with a magnitude of about 10^{-4} to 10^{-5}, for a confined aquifer. Since decreasing S in Eq.(1.12) causes $\partial s / \partial t$ to increase, this large difference in magnitude for S in confined and unconfined aquifers has a profound influence upon drawdown response times in these two different kinds of aquifers.

Eq.(1.12) is classified mathematically as a second-order equation since the order of a differential equation is fixed by the highest order derivative. All second-order equations are further classified as elliptic, parabolic or hyperbolic. This terminology originates from an analogy in analytic geometry, and the classification of any second-order equation is found by determining the number of real "characteristics" for the equation. Usually, steady-state processes are described by elliptic equations, and unsteady processes are described by either parabolic or hyperbolic equations. (However, important exceptions to this occur. In the boundary-layer problem of fluid mechanics, the steady-flow form of Prandtl's simplified boundary-layer equations are parabolic. Furthermore, an important class of unsteady problems concerned with the movement of waves over the free surface of a body of water are described by the Laplace equation, which is elliptic.) Parabolic equations describe diffusion processes, in which oscillations cannot be expected to occur unless a forcing function contains oscillations, and hyperbolic equations describe wave mechanics problems, in which oscillations can be expected to occur for almost any form of a forcing function. A very well known mathematician once noted that all hyperbolic equations describe the movement of waves, but not all wave mechanics problems are described by hyperbolic equations. (As noted above, some free-surface wave mechanics problems are described by an elliptic equation.) Eq.(1.12) is a parabolic equation when the time derivative on the right side does not vanish (unsteady flow). On the other hand, this time derivative vanishes for steady flow, and then the equation becomes elliptic. If all of this confuses you, don't worry about it. We will do nothing with the classification of second-order equations beyond noting that such a classification exists.

It is worth pointing out that Eq.(1.12) is found in a number of application areas that are of interest to civil engineers. For example, an equation of this form describes the unsteady flow of heat through either a solid or motionless fluid, where Darcy's law is replaced with Fourier's law of heat conduction. Therefore, Eq.(1.12) is sometimes called the unsteady heat conduction equation. Eq.(1.12) is also encountered when studying contaminant diffusion in a motionless body of fluid. In this case, Darcy's law is replaced with Fick's first law of diffusion, and Eq.(1.12) is called the diffusion equation. Thus, Eq.(1.12) provides a good example of how the study of a mathematical equation in one application area can provide insight into the behaviour of phenomena observed in a number of different areas. It also illustrates something which, by now, must be obvious to many of you: mathematics is one of the most generally applicable and useful subjects that any of us will ever study. It is our hope that you also find it an interesting subject to study.

Lecture 2

A Typical Problem and Dimensionless Variables

One of the most important skills that any analyst must develop is the ability to make a complete and concise statement of the equations that must be solved for a given problem. Eq.(1.12) contains second-order spatial derivatives but only a first-order time derivative. This means that s, the derivative of s or a combination of the two must be specified along each and every spatial boundary of the solution domain, and the presence of the first-order time derivative means that one initial condition must be specified at t = 0.

For an example, consider the problem shown in Fig. 2.1. A river flood plain, with an underlying sand and gravel aquifer, is bounded on the left by a long straight river edge and on the right by a long straight clay embankment that parallels the river edge. The clay embankment is impermeable, and the river level remains constant as irrigation water is spread uniformly, at a constant rate, R, over the entire flood plain for $0 < t < \infty$. The entire system is assumed to start from a state of rest at t = 0, when water table levels everywhere in the sand and gravel aquifer have the same constant elevation as the river level.

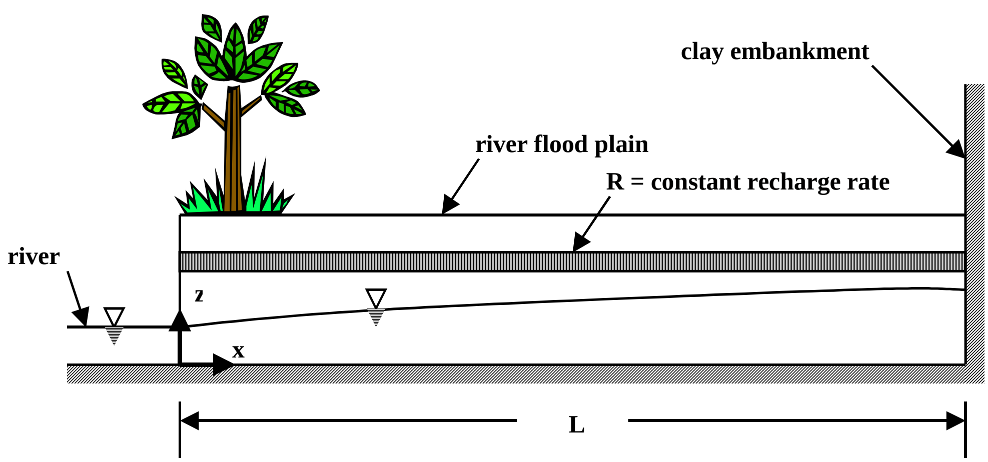

Fig. 2.1. Groundwater recharge to an aquifer beneath a river flood plain.

The set of equations used to model the problem in Fig. 2.1 must include a partial differential equation that holds in all parts of the solution domain, $0 < x < L$ and $0 < t < \infty$, where x is a horizontal coordinate measured as positive away from the river edge. Since the river and clay embankment are assumed to be parallel and to extend to $\pm\infty$, the operator $\nabla^2 s \equiv \partial^2 s / \partial x^2 + \partial^2 s / \partial y^2$ in Eq.(1.12) reduces to $\nabla^2 s \equiv \partial^2 s / \partial x^2$ [since s does not change with y when x and t are held constant. i.e. s = s(x, t)]. One boundary condition must be specified at x = 0, and a second boundary condition must be specified at x = L. Finally, because of the presence of $\partial s / \partial t$ in Eq.(1.12), an initial condition must also be specified at t = 0. Thus, a complete mathematical statement of this problem is given by the following equations:

$$T \frac{\partial^2 s}{\partial x^2} = S \frac{\partial s}{\partial t} + R \qquad (0 < x < L,\ 0 < t < \infty) \tag{2.1}$$

$$s(0, t) = 0 \qquad (0 < t < \infty) \tag{2.2}$$

$$\frac{\partial s(L, t)}{\partial x} = 0 \qquad (0 < t < \infty) \tag{2.3}$$

$$s(x, 0) = 0 \qquad (0 < x < L) \tag{2.4}$$

where R is a constant. Notice that specified ranges for independent variables are given in brackets at the right of each equation. Not everyone follows this procedure, but anyone who either solves this problem or who makes use of a solution obtained by someone else must be aware of this information, and it makes life easier for all concerned if these ranges are given in the problem statement. I expect you to do this as well!

Eq.(2.1) is the partial differential equation that Eq.(1.12) reduces to under the circumstances depicted in Fig. 2.1. Eq.(2.2) is a boundary condition that requires the river level to remain fixed at its initial level for $0 < t < \infty$. Eq.(2.3) is a boundary condition that uses Darcy's law, in the form of either Eq.(1.1) or (1.3), to require that the clay embankment at x = L be impermeable. Eq.(2.4) is an initial condition that requires the free surface in the aquifer to have the same constant elevation as the river level at t = 0.

There are two mathematical questions that you should be aware of, even though we will do nothing in this course to formally answer them. The first question concerns uniqueness, which shows that all solutions of Eqs.(2.1)-(2.4) give the same numerical values for s(x, t) at every point within the solution domain. This is important since there are numerous ways in which a solution of these equations can be found, and solutions found by these different ways often do not have identical forms. It is also important because it gives a good indication that the correct number and type of equations have been given in the problem statement. Uniqueness for Eqs.(2.1)-(2.4) is fairly easy to prove, but we will not do so. Uniqueness for more difficult equations, though, is not always easy to prove. For example, the Navier-Stokes equations of fluid mechanics are known from experiment to sometimes have at least two solutions, one for laminar flow and one for turbulent flow, at the same Reynolds number, and whether one or the other of these will occur depends upon whether a disturbance of the correct magnitude and frequency is present in the flow.

The second mathematical question concerns existence: does a solution of these equations actually exist? This question is relatively difficult to answer and is normally considered only by very good mathematicians. Existence is shown by outlining a way in which a solution can be calculated. This solution method need not be, and frequently is not, a practical way in which to actually calculate a solution. Sometimes, however, existence proofs lead to practical methods of calculation. A good example of this is provided by the use of singular integral equations to solve irrotational flow problems in fluid mechanics. These integral equations were used by mathematicians to prove existence for irrotational flow problems many years before computers became available. When computers became available in the early 1960’s, this method turned out to be the most accurate and efficient way available for calculating numerical solutions of the Laplace equation. In engineering circles it now masquerades under the name “boundary-element method.”

The practical calculation of a solution, and its numerical evaluation, is invariably simplified by introducing dimensionless variables. We will do this for Eqs.(2.1)-(2.4) by scaling x with the distance L in Fig. 2.1. However, suitable scales for s and t are not so obvious. Therefore, we will use the constants s_0 and t_0, respectively, for these scales and then choose s_0 and t_0 so that as many coefficients as possible in the resulting equations are unity. This means that we start by inserting in Eqs.(2.1)-(2.4) the following dimensionless variables:

$$\left(s^*, x^*, t^*\right) = \left(\frac{s}{s_0}, \frac{x}{L}, \frac{t}{t_0}\right) \tag{2.5}$$

The formal introduction of these dimensionless variables is carried out by using the chain rule of differential calculus.

$$\frac{\partial s}{\partial x} = \frac{\partial (s_0 s^*)}{\partial x^*}\frac{\partial x^*}{\partial x} = s_0 \frac{\partial s^*}{\partial x^*}\frac{1}{L} = \frac{s_0}{L}\frac{\partial s^*}{\partial x^*} \tag{2.6}$$

$$\frac{\partial^2 s}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial s}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{s_0}{L}\frac{\partial s^*}{\partial x^*}\right) = \frac{\partial}{\partial x^*}\left(\frac{s_0}{L}\frac{\partial s^*}{\partial x^*}\right)\frac{\partial x^*}{\partial x} = \frac{s_0}{L^2}\frac{\partial^2 s^*}{\partial x^{*2}} \tag{2.7}$$

A much faster way of doing the same operation is to realize that since s, x and t are scaled with s_0, L and t_0, respectively, we have

$$\frac{\partial s}{\partial x} \sim \frac{s_0}{L} \tag{2.8}$$

$$\frac{\partial^2 s}{\partial x^2} \sim \frac{s_0}{L^2} \tag{2.9}$$

Regardless of how we carry out this operation, the coefficients of the three terms in Eq.(2.1) are found to be

$$T\frac{s_0}{L^2} \qquad S\frac{s_0}{t_0} \qquad R \tag{2.10}$$

Make the first coefficient in Eq.(2.10) unity by division with the first term.

$$1 \qquad \frac{SL^2}{Tt_0} \qquad \frac{RL^2}{Ts_0} \tag{2.11}$$

Eq.(2.11) now shows that the second and third coefficients will also be unity if we make the following choices for t_0 and s_0:

$$\left(t_0, s_0\right) = \left(\frac{SL^2}{T}, \frac{RL^2}{T}\right) \tag{2.12}$$

Thus, we have arrived at the following set of dimensionless variables to replace Eq.(2.5):

$$\left(s^*, x^*, t^*\right) = \left(\frac{sT}{RL^2}, \frac{x}{L}, \frac{tT}{SL^2}\right) \tag{2.13}$$

Furthermore, we now know that Eq.(2.1) simplifies under this transformation to

$$\frac{\partial^2 s^*}{\partial x^{*2}} = \frac{\partial s^*}{\partial t^*} + 1 \qquad \left(0 < x^* < 1,\ 0 < t^* < \infty\right) \tag{2.14}$$

With a little practice, you will soon learn to carry out this operation very efficiently. In fact, it is identical with the method that you will probably see used in Fluid Mechanics 2 when obtaining order-of-magnitude estimates for terms in the Navier-Stokes equations.

There is still one little detail to attend to before we write down the final set of dimensionless equations that will replace Eqs.(2.1)-(2.4). It is cumbersome and time consuming to carry the asterisk superscript along in our notation for dimensionless variables. Therefore, we will omit this superscript in our equations, but we will do so with the understanding that the symbols (s, x, t) are actually an abbreviated notation for (s^*, x^*, t^*) in

Eq.(2.13). We usually do this by stating breezily that "the asterisk superscript has been omitted for notational convenience." This may seem confusing at first. However, it makes a real simplification in the calculations, and a little practice soon makes it very easy to implement. After these preliminaries, the final form for our problem statement becomes

$$\frac{\partial^2 s}{\partial x^2} = \frac{\partial s}{\partial t} + 1 \qquad \left(0 < x < 1,\ 0 < t < \infty\right) \tag{2.15}$$

$$s\left(0, t\right) = 0 \qquad \left(0 < t < \infty\right) \tag{2.16}$$

$$\frac{\partial s(1, t)}{\partial x} = 0 \qquad \left(0 < t < \infty\right) \tag{2.17}$$

$$s\left(x, 0\right) = 0 \qquad \left(0 < x < 1\right) \tag{2.18}$$

in which all variables are dimensionless and are defined in Eq.(2.13). We will solve this set of equations in the fourth lecture.

Lecture 3

Orthogonality Relationships and Fourier Series

Many of the problems solved in this set of lectures will make use of a Fourier series in one of the two following forms:

$$s(x,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t) \cos\left(\alpha_n \frac{x}{L}\right) \qquad (0 < x < L,\ 0 < t < \infty) \qquad (3.1)$$

$$s(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\alpha_n \frac{x}{L}\right) \qquad (0 < x < L,\ 0 < t < \infty) \qquad (3.2)$$

These infinite series are known as "half-range Fourier series." Eq.(3.1) is used when the boundary condition $\partial s(0,t)/\partial x = 0$ applies, and Eq.(3.2) is used when the boundary condition $s(0,t) = 0$ applies. Then, in both cases, we will choose the "eigenvalue" α_n so that either $\partial s(L,t)/\partial x = 0$ or $s(L,t) = 0$. **These choices will be made so that the chosen series expansion satisfies exactly the specified homogeneous boundary conditions for s(x, t) at both x = 0 and x = L.**[2]

At one or more points in our analysis we will need to calculate the coefficients, $a_n(t)$ or $a_n(0)$, in our chosen infinite series. We will always do this by making use of what are called "orthogonality relationships" for the Fourier series. In this context, the term orthogonality arises not because terms are in some way perpendicular to each other but because of an analogous procedure that is often used in vector analysis to solve for vector components when a vector is written in terms of a set of orthogonal unit base vectors. For example, if $\hat{i}$, $\hat{j}$ and $\hat{k}$ are orthogonal unit base vectors in a Cartesian coordinate system, and if it is wanted to write a known vector, $\overline{V}$, as follows:

$$\overline{V} = a\hat{i} + b\hat{j} + c\hat{k} \qquad (3.3)$$

where a, b and c are unknown, then forming the dot product of $\overline{V}$ with each of the three orthogonal unit base vectors gives

$$\overline{V} \cdot \hat{i} = a \qquad (3.4)$$

$$\overline{V} \cdot \hat{j} = b \qquad (3.5)$$

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[2] It is also possible to consider a third type of boundary condition, $\partial s / \partial x + \lambda s = 0$, but we will not do so.

$$\overline{V}\cdot\hat{k}=c \tag{3.6}$$

where use has been made of the orthogonality relationships $\hat{i}\cdot\hat{j}=0$, $\hat{i}\cdot\hat{k}=0$ and $\hat{j}\cdot\hat{k}=0$.

The analogous procedure for solving Eq.(3.1) or (3.2) for a_n makes use of the following orthogonality relationships:

$$\int_0^L \cos\left(\alpha_m \frac{x}{L}\right)\ \cos\left(\alpha_n \frac{x}{L}\right) dx = 0 \qquad (m \neq n) \tag{3.7}$$

$$\int_0^L \sin\left(\alpha_m \frac{x}{L}\right)\ \sin\left(\alpha_n \frac{x}{L}\right) dx = 0 \qquad (m \neq n) \tag{3.8}$$

$$\int_0^L \cos\left(\alpha_n \frac{x}{L}\right) dx = 0 \qquad (n \geq 1, \text{ only if } \alpha_n = n\pi) \tag{3.9}$$

In addition, we will need to make use of the following integrals:

$$\int_0^L \sin^2\left(\alpha_n \frac{x}{L}\right) dx = \int_0^L \cos^2\left(\alpha_n \frac{x}{L}\right) dx = \frac{L}{2} \quad (n \geq 1) \tag{3.10}$$

$$\int_0^L dx = L \tag{3.11}$$

As we will show later, it is necessary to assume that α_n has been chosen in these equations so that either $\cos(\alpha_n x/L)$ or $\sin(\alpha_n x/L)$, or else the first derivative of these functions, vanishes at x = L. This will always be the case if our Fourier series satisfies the homogeneous boundary conditions either $s=0$ or $\partial s/\partial x = 0$ at both x = 0 and x = L.

For a specific example, suppose that we want to represent a known function, f(x), with the following infinite series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\alpha_n \frac{x}{L}\right) \qquad (0 < x < L) \tag{3.12}$$

where the coefficients a_n are unknown. [This series representation can be used only if $f'(0)=0$ and if a_0 and α_n have been chosen so that either $f'(L)=0$ or $f(L)=0$.[3] Then we

[3] If $f'(L)=0$, then $a_0 \neq 0$ and $\alpha_n = n\pi$ in Eq.(3.12). If $f(L)=0$, then $a_0=0$ and $\alpha_n = (2n-1)\pi/2$ in Eq.(3.12).

can make use of Eqs.(3.7)-(3.11) to calculate a_n .] Integrate both sides of Eq.(3.12) from x = 0 to x = L and make use of Eqs.(3.9) and (3.11) to obtain an expression for a_0.

$$\int_0^L f(x)\, dx = a_0 L + 0 + 0 + 0 + \cdots \tag{3.13}$$

Next, multiply both sides of Eq.(3.12) by $\cos(\alpha_m x / L)$ to obtain

$$f(x)\cos\left(\alpha_m \frac{x}{L}\right) = a_0 \cos\left(\alpha_m \frac{x}{L}\right) + a_1 \cos\left(\alpha_1 \frac{x}{L}\right)\cos\left(\alpha_m \frac{x}{L}\right) + a_2 \cos\left(\alpha_2 \frac{x}{L}\right)\cos\left(\alpha_m \frac{x}{L}\right) + \cdots$$
$$+ a_m \cos\left(\alpha_m \frac{x}{L}\right)\cos\left(\alpha_m \frac{x}{L}\right) + \cdots$$
$$\tag{3.14}$$

Then integrate both sides of Eq.(3.14) from x = 0 to x = L and make use of Eqs.(3.7), (3.9) and (3.10) to obtain

$$\int_0^L f(x)\cos\left(\alpha_m \frac{x}{L}\right) dx = 0 + 0 + 0 + \cdots + a_m \frac{L}{2} + \cdots \tag{3.15}$$

Thus, we have obtained the following expressions for a_m :

$$a_0 = \frac{1}{L}\int_0^L f(x)\, dx \tag{3.16}$$

$$a_m = \frac{2}{L}\int_0^L f(x)\cos\left(\alpha_m \frac{x}{L}\right) dx \qquad (m \geq 1) \tag{3.17}$$

This process of using orthogonality conditions to “pick off” the Fourier coefficients is extremely important, and we will need to use it , sometimes several times, when using Fourier series to solve any of our problems.

We will prove the relationships given in Eqs.(3.7)-(3.10) by letting $y_n(x)$ represent term n in any one of the following three sets of functions:

$$1, \cos\left(\alpha_1 \frac{x}{L}\right), \cos\left(\alpha_2 \frac{x}{L}\right), \cos\left(\alpha_3 \frac{x}{L}\right), \cdots \tag{3.18}$$

$$\cos\left(\alpha_1 \frac{x}{L}\right), \cos\left(\alpha_2 \frac{x}{L}\right), \cos\left(\alpha_3 \frac{x}{L}\right), \cdots \tag{3.19}$$

$$\sin\left(\alpha_1 \frac{x}{L}\right), \sin\left(\alpha_2 \frac{x}{L}\right), \sin\left(\alpha_3 \frac{x}{L}\right), \cdots \tag{3.20}$$

If $y_n(x)$ is a term in the set (3.18), then $y'(0) = 0$ and α_n has been chosen so that $y'(L) = 0$. If $y_n(x)$ is a term in the set(3.19), then $y'(0) = 0$ and α_n has been chosen so that

$y(L)=0$. If $y_n(x)$ is a term in the set (3.20), then $y(0)=0$ and α_n has been chosen so that either $y(L)=0$ or $y'(L)=0$. Therefore, regardless of whether $y_n(x)$ represents term n in set (3.18), (3.19) or (3.20), the following equation applies:

$$y_n(x)\, y_n'(x)=0 \qquad (\text{at both } x=0 \text{ and } x=L) \tag{3.21}$$

Furthermore, $y_n(x)$ is a solution of the following ordinary differential equation:

$$y_n'' + (\alpha_n / L)^2\, y_n = 0 \qquad (0<x<L,\ \alpha_0 \equiv 0) \tag{3.22}$$

Multiply Eq.(3.22) by y_m to obtain

$$y_m y_n'' + (\alpha_n / L)^2\, y_m y_n = 0 \tag{3.23}$$

A second equation can be obtained from Eq.(3.23) by interchanging the subscripts m and n.

$$y_n y_m'' + (\alpha_m / L)^2\, y_n y_m = 0 \tag{3.24}$$

Subtract Eq.(3.24) from Eq.(3.23).

$$y_m y_n'' - y_n y_m'' + \frac{\left(\alpha_n^2 - \alpha_m^2\right)}{L^2} y_m y_n = 0 \tag{3.25}$$

However, the first two terms in Eq.(3.25) can be rewritten to obtain an equivalent equation.

$$\left(y_m y_n' - y_n y_m'\right)' + \frac{\left(\alpha_n^2 - \alpha_m^2\right)}{L^2} y_m y_n = 0 \tag{3.26}$$

[If you doubt Eq.(3.26), differentiate the difference of the two terms in brackets and show that this gives Eq.(3.25).] Finally, integrate Eq.(3.26) from x = 0 to x = L to obtain

$$\left(y_m y_n' - y_n y_m'\right)_{x=0}^{x=L} + \frac{\left(\alpha_n^2 - \alpha_m^2\right)}{L^2} \int_0^L y_m(x)\, y_n(x)\, dx = 0 \tag{3.27}$$

The first term in brackets vanishes at both the lower and upper limits of integration by virtue of Eq.(3.21), and, since $\left(\alpha_n^2 - \alpha_m^2\right) \neq 0$ when $m \neq n$, we are left with final result

$$\int_0^L y_m(x)\, y_n(x)\, dx = 0 \qquad (m \neq n) \tag{3.28}$$

Eq.(3.28) is identical with the orthogonality relationships given by Eqs.(3.7)-(3.9). This proof and final result is a specific application of a more general proof that is given in texts under the heading of Sturm-Liouville theory. [For example, Hildebrand (1965)]

Proof of Eq.(3.10) is easily carried out by setting m = n in Eq.(3.23) to obtain

$$y_n\, y_n'' + (\alpha_n / L)^2\, y_n^2 \equiv \left(y_n\, y_n'\right)' - \left(y_n'\right)^2 + (\alpha_n / L)^2\, y_n^2 = 0 \tag{3.29}$$

Multiplication of Eq.(3.22) by $y_n^{'}$ gives the result

$$y_n^{'}\; y_n^{''} + \left(\alpha_n / L\right)^2\; y_n^{'}\; y_n \equiv \frac{1}{2}\frac{d}{dx}\left[\left(y_n^{'}\right)^2 + \left(\alpha_n / L\right)^2\; y_n^2\right] = 0 \qquad (3.30)$$

Thus, integration of Eq.(3.30) gives

$$\left(y_n^{'}\right)^2 + \left(\alpha_n / L\right)^2\; y_n^2 = C = \left(\alpha_n / L\right)^2 \qquad (3.31)$$

where the integration constant, C, has been evaluated at x = 0. (Either $y_n^{'} = 0$ and $y_n = 1$ or else $y_n^{'} = \alpha_n / L$ and $y_n = 0$ at x = 0.) Substituting for $\left(y_n^{'}\right)^2$ from Eq.(3.31) in Eq.(3.29) gives

$$\left(y_n\; y_n^{'}\right)^{'} - \left(\alpha_n / L\right)^2 + 2\left(\alpha_n / L\right)^2\; y_n^2 = 0 \qquad (3.32)$$

Then integrating Eq.(3.32) from x = 0 to x = L and making use of Eq.(3.21) leads to the final result.

$$\int_0^L y_n^2\left(x\right)\, dx = \frac{L}{2} \qquad \left(n \geq 1\right) \qquad (3.33)$$

Eq.(3.33) is identical with Eq.(3.10).

Lecture 4
Solution of the Problem Shown in Fig. 2.1

You have been accustomed to solving ordinary differential equations by obtaining the sum of a general solution and a particular solution and then using either initial or boundary conditions to determine unknown constants in the general solution. The general solution of a partial differential equation, however, contains unknown functions with specified arguments, and the determination of these unknown functions is not always easy. Furthermore, general solutions of second-order partial differential equations can be found only for a small number of the very simplest equations. As a result, solution of problems involving partial differential equations necessarily proceeds on a case by case basis, in which solution methods that work for one problem may not work for slightly different problems.

The traditional starting point for most courses at this level is to introduce a method known as "separation of variables." Examples of this method are given in almost any book on advanced engineering mathematics, including the book by Zill and Cullen that you used in your first-professional math course. However, we will use a slightly more general method, one that Hildebrand (1976) has called "the method of variation of parameters." Like the method of separation of variables, this method works only for systems of linear equations with constant coefficients, only spatial derivatives of even order can appear in the partial differential equation and convergence of the infinite series solution is most rapid if boundary conditions are homogeneous. On the other hand, this method routinely deals with more general partial differential equations and initial conditions than can solved by separating variables, and I believe that it is easier to learn.

In this lecture we will use variation of parameters to solve the following problem, which was pictured in Fig. 2.1 and described by Eqs.(2.15)-(2.18):

$$\frac{\partial^2 s}{\partial x^2} = \frac{\partial s}{\partial t} + 1 \qquad \left(0 < x < 1,\ 0 < t < \infty\right) \tag{4.1}$$

$$s(0,t)=0 \qquad (0<t<\infty) \tag{4.2}$$

$$\frac{\partial s(1,t)}{\partial x}=0 \qquad (0<t<\infty) \tag{4.3}$$

$$s(x,0)=0 \qquad (0<x<1) \tag{4.4}$$

We start by looking for a solution of Eq.(4.1) in the form of a half-range Fourier series that satisfies exactly the homogeneous boundary conditions (4.2) and (4.3).[4]

$$s(x,t)=\sum_{n=1}^{\infty} a_n(t)\,\sin(\alpha_n x) \qquad (0<x<1,\ 0<t<\infty) \tag{4.5}$$

where the constant α_n is given by the odd multiples of $\pi/2$.

$$\alpha_n=(2n-1)\frac{\pi}{2} \quad (n=1,\ 2,\ 3,\ \cdots) \tag{4.6}$$

The boundary condition at x = 0 determined that a sin series must be used in Eq.(4.5), and the boundary condition at x = 1 determined values for α_n. This will always be the case: the homogeneous boundary condition at x = 0 determines whether we use a series of the form given by Eq.(3.1) or by Eq.(3.2), and the homogeneous boundary condition at x = L fixes a value for α_n, which is either $n\pi$ or $(2n-1)\pi/2$. In this problem, of course, L = 1.

Eq.(4.5) satisfies Eqs.(4.2) and (4.3). Substituting Eq.(4.5) into Eq.(4.1) shows what is required in order that the partial differential be satisfied as well.

$$-\sum_{n=1}^{\infty}\alpha_n^2\ a_n(t)\sin(\alpha_n x)=\sum_{n=1}^{\infty}\frac{da_n(t)}{dt}\sin(\alpha_n x)\ +1 \tag{4.7}$$

Since $\sin(\alpha_n x)$ satisfies the conditions required for the validity of the results given by Eqs.(3.7)-(3.10) (i.e. It vanishes at x = 0, and its first derivative vanishes at x = 1), we can multiply Eq.(4.7) by $\sin(\alpha_m x)$ and integrate from x = 0 to x = 1 to obtain a differential equation for $a_n(t)$.

[4] Boundary conditions are conditions composed on the spatial boundary of the solution domain, in this case on x = 0 and x = 1. These boundary conditions are homogeneous because s = 0 satisfies them, even though s = 0 does not satisfy all of the remaining equations in the problem statement. This definition of homogeneity means that Eqs.(4.2), (4.3) and (4.4) are all homogeneous, and Eq.(4.1) is said to be either nonhomogeneous or inhomogeneous since s = 0 does not reduce it to an identity.

$$-\alpha_m^2 \; a_m(t)\,\frac{1}{2} = \frac{da_m(t)}{dt}\,\frac{1}{2} + \frac{1}{\alpha_m} \qquad (4.8)$$

Eq.(4.8) is a first-order ordinary differential equation that requires one initial condition at t = 0 for its unique solution. By considering both Eqs.(4.4) and (4.5), we see that the initial condition for $s(x,0)$ will be satisfied if we choose to set

$$a_m(0) = 0 \qquad (4.9)$$

Alternatively, we could insert Eq.(4.5) into Eq.(4.4) and use Eqs.(3.7)-(3.10) to show that Eq.(4.9) is necessary in order that these two equations be satisfied. Thus, although either approach can be used to arrive at Eq.(4.9), we see that Eq.(4.9) is both necessary and sufficient if Eq.(4.4) is to be satisfied by Eq.(4.5). The solution of Eqs.(4.8) and (4.9) is given by

$$a_m(t) = 2\,\frac{e^{-t\alpha_m^2} - 1}{\alpha_m^3} \qquad (4.10)$$

Thus, changing m to n in Eq.(4.10) and inserting the result in Eq.(4.5) gives the final solution of our problem.

$$s(x,t) = -2\sum_{n=1}^{\infty} \frac{\left(1 - e^{-t\alpha_n^2}\right)}{\alpha_n^3}\,\sin(\alpha_n x) \qquad \left(0 < x < 1,\; 0 < t < \infty\right) \qquad (4.11)$$

where $\alpha_n = (2n-1)\pi/2$. Values of $s(x,t)$ calculated from Eq.(4.11) will be negative because a drawdown is positive when the free surface drops, and, in this aquifer recharge problem, the free surface rises.

Lecture 5

Plotting the Solution Given by Eq. (4.11)

It is important to learn how to calculate a solution, but it is also important to know how to plot it to obtain useful results. Therefore, we will write a user-defined function in Visual Basic for Applications that evaluates s(x,t) from Eq.(4.11). Then we will use this program to plot results for a specific example. Finally, since the governing equations are linear and have time-independent coefficients, we will use the principles of superposition and time translation to calculate and plot a solution when the recharge rate, R, has a relatively simple variation with time.

The following solution is given by Eq.(4.11):

$$\begin{aligned} |s(x,t)| &= 2\sum_{n=1}^{\infty}\frac{\left(1-e^{-t\alpha_n^2}\right)}{\alpha_n^3}\sin(\alpha_n x) \qquad \left[0<x<1,\ 0\le t<\infty,\ \alpha_n=(2\pi-1)\frac{\pi}{2}\right] \\ &= 0 \qquad (0<x<1,\ -\infty<t\le 0) \end{aligned} \tag{5.1}$$

It is important to remember that s, x and t in Eq.(5.1) actually represent variables with an asterisk superscript given by Eq.(2.13).

$$(s^*, x^*, t^*) = \left(\frac{sT}{RL^2}, \frac{x}{L}, \frac{tT}{SL^2}\right) \tag{5.2}$$

We will write our Visual basic program with the notation shown in Eq.(5.1), which makes the programming easier, but we will have to use Eq.(5.2) when the program is used to calculate numbers in a spreadsheet. People who want to review spreadsheet calculations and program construction for user-defined functions are advised to read any edition of the outstanding introduction by Liengme (2000).

The programming process is started by opening a spreadsheet and clicking on **Tools—Macro—Visual Basic Editor**. This opens a page in the editor, at which point click on **Insert—Module** to open a page upon which the following program is typed:

```
'The following program calculates the rise in a groundwater free surface
'by evaluating Eq.(5.1). All input and output variables are dimensionless
'and are defined as follows:
'   s*=abs(s)T/RL^2       x*=x/L       t*=tT/SL^2
'where s=drawdown, T=transmissivity, R=recharge velocity, L=aquifer length,
'x=distance from the river edge, t=time and S=porosity. nmax=number of terms
'used in the summation. All calculations are carried out using dimensionless
'variables, but the asterisk superscript has been omitted for notational convenience.
Function s(x, t, nmax)
If t <= 0 Then
   s = 0#
Else
   Pi = 3.141592654
   s = 0#
   For n = 1 To nmax
      alpha = (2 * n - 1) * Pi / 2
      term = 2 * (1 - Exp(-t * alpha ^ 2)) * Sin(alpha * x) / alpha ^ 3
      s = s + term
   Next n
End If
End Function
```

Comment statements are preceded by ' and appear at the start of the program. They are very important since they tell the user what the program calculates and define variables used in the program. Setting t = 0 when $t \le 0$ is essential for some of the applications that follow. The numerical value of pi was obtained by entering =PI() in a spreadsheet cell. Then the resulting numerical value of pi was copied and pasted into the program in the module. The variable **nmax** is used to control the number of terms used in the series expansion for $t > 0$. There are other ways to accomplish this result, such as using a "Do Until" loop. However, if you choose this option, be careful! Values of "Term" in the program can have a small absolute value for some values of n when either **(1 - Exp(-t * alpha ^ 2))** or **Sin(alpha * x)** are small, but **term** may become larger again as n increases. In other words, make sure that you do not terminate the series prematurely. Probably the best way to choose **nmax** is to make use of the inequality

$$\left| 2\frac{\left(1-e^{-t\alpha_n^2}\right)}{\alpha_n^3}\sin\left(\alpha_n x\right)\right| \le 2\frac{\left|1-e^{-t\alpha_n^2}\right|}{\left|\alpha_n^3\right|}\left|\sin\left(\alpha_n x\right)\right| \le \frac{2}{\alpha_n^3} \qquad (5.3)$$

Thus, if we choose **nmax** to be 15, then $1/\alpha_{n\max}^{3} = 0.000000232$ and each of the neglected terms, when n > **nmax**, will have a magnitude that is less than 0.000000232.

The result of using this program to plot the solution for a particular problem is shown in Fig. 5.1. Variables and parameters that are held constant during the calculations are entered in rows one and two. (It is very important to include the dimensions for each variable and parameter.) The connection between the dimensionless and dimensional variables, given in Eq.(5.2), is made in the spreadsheet by entering in cell B7 the formula

=($A$2*$B$2^2/$C$2)*s($A7/$B$2,B$4*$C$2/($D$2*$B$2^2),$E$2)

Notice the way in which absolute and relative addressing have been used so that this formula can be dragged across and down. An alternative way to do this is to create names for variables, which allows the user to type symbols instead of cell addresses into the formula. However, I suspect that it would still be necessary to use absolute and relative addresses when entering values for t across row four since naming a variable gives it an absolute cell address. (i.e. The column letter and row number are **both** preceded by $.)

	A	B	C	D	E	F	G	H
1	R (m/day)	L (m)	T (m^2/day)	S	nmax			
2	0.001	500	2000	0.07	15			
3								
4	t (days) =	1	2.5	5	15			
5								
6	x (m)	Values of Abs(s) in m						
7	0	0	0	0	0			
8	10	0.00091523	0.00146	0.001967	0.002432			
9	20	0.00180423	0.002893	0.003906	0.004836			
10	30	0.00264821	0.004279	0.005798	0.007191			
11	40	0.00343962	0.005611	0.007633	0.00949			
12								
13								
14								
15								
16								
17								
18								
19								
20								
21								
22								
23								
24								
25								
26								
27								
28	210	0.0114804	0.021935	0.031821	0.040898			
29	220	0.01172212	0.02258	0.032862	0.042302			
30	230	0.01194551	0.023196	0.033862	0.043656			
31	240	0.01215065	0.02378	0.034821	0.044959			
32	250	0.01233874	0.024335	0.035739	0.046211			
33	260	0.01251191	0.024862	0.036618	0.047414			

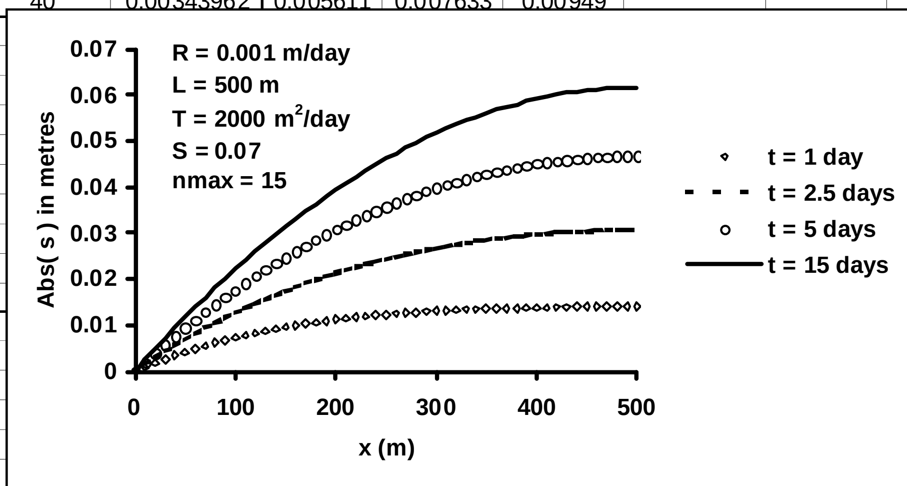


Fig. 5.1. A spreadsheet program that plots the solution given by Eq.(5.1).

All of the governing equations, in this case Eqs.(4.1)-(4.4), are linear with coefficients that do not depend upon t. This means that superposition and time translation can be used to calculate the solution for a "pulse." A pulse occurs when the recharge velocity vanishes for $-\infty < t < 0$ and $\Delta < t < \infty$ but equals a constant, R, for $0 < t < \Delta$. In this case, the solution for a pulse, $s_p(x,t)$, satisfies Eqs.(4.2)-(4.4) and the following modified form of Eq.(4.1):

$$\begin{aligned} \frac{\partial^2 s_p}{\partial x^2} - \frac{\partial s_p}{\partial t} &= 1 \qquad (0 < x < 1,\ 0 < t < \Delta) \\ &= 0 \qquad (0 < x < 1,\ -\infty < t < 0 \text{ and } \Delta < t < \infty) \end{aligned} \tag{5.4}$$

However, the solution given by Eq.(5.1), $s(x,t)$, satisfies

$$\frac{\partial^2 s}{\partial x^2} - \frac{\partial s}{\partial t} = 1 \quad (0 < x < 1,\ 0 < t < \infty)$$
$$= 0 \quad (0 < x < 1,\ -\infty < t < 0) \tag{5.5}$$

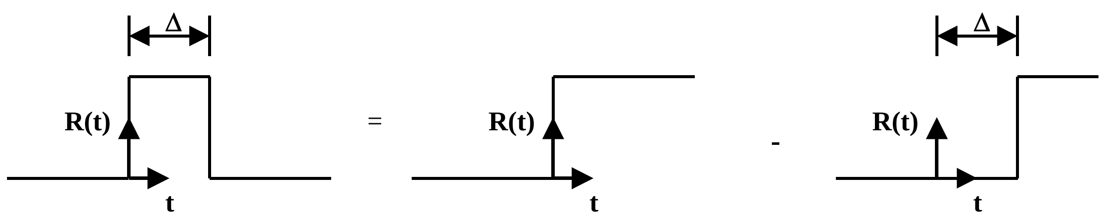
Furthermore, the time-translation principle just mentioned means that $s(x, t-\Delta)$ satisfies an equation similar to Eq.(5.5) but with a translation in t of Δ.

$$\frac{\partial^2 s}{\partial x^2} - \frac{\partial s}{\partial t} = 1 \quad (0 < x < 1,\ \Delta < t < \infty)$$
$$= 0 \quad (0 < x < 1,\ -\infty < t < \Delta) \tag{5.6}$$

Thus, direct substitution into Eq.(5.4) and use of Eqs.(5.5) and (5.6) shows that

$$s_p(x,t) = s(x,t) - s(x, t-\Delta) \qquad (0 < x < 1,\ -\infty < t < \infty) \tag{5.7}$$

is the solution of Eq.(5.4) and Eqs.(4.2)-(4.4). It is easy to remember this result if you keep in mind the following sketch:



(If this looks familiar, it may be because exactly the same kind of manipulation is used in unit hydrograph theory when using an "S curve" to obtain a river hydrograph created by a rainfall pulse on a watershed catchment.)

Since $s(x,t)$ vanishes when t < 0, an alternative but longer way to write Eq.(5.7) is

$$s_p(x,t) = 0 \qquad (0 < x < 1,\ -\infty < t \le 0)$$
$$= s(x,t) \qquad (0 < x < 1,\ 0 \le t \le \Delta) \tag{5.8}$$
$$= s(x,t) - s(x,t-\Delta) \qquad (0 < x < 1,\ \Delta < t < \infty)$$

However, if the program that calculates $s(x,t)$ sets $s(x,t) = 0$ when t < 0 (as our program does), then the following very short program is sufficient to calculate the solution for a pulse that starts at t = 0, has a height R (which is incorporated in the dimensionless variable s*) and extends to t = Δ:

```
'The following program calculates the rise in a groundwater free surface
'caused by a recharge pulse. All variables have been defined in the
'user-defined function s(x, t, nmax). In addition, the dimensionless
'pulse time duration is defined as
'                    delta*=delta T/SL^2
Function s_pulse(x, t, delta, nmax)
s_pulse = s(x, t, nmax) - s(x, t - delta, nmax)
End Function
```

The result of applying this program to the problem under consideration is shown in Fig. 5.2. It was found that **nmax** had to be increased from the value of 15 that was used in Fig. 5.1. (Values of 15 for **nmax** were too small and created inaccuracies that were evidenced by oscillations in the plotted curves. This is because values of s(x,t) are considerably smaller in Fig. 5.2, and a truncation error that was acceptable for the example shown in Fig. 5.1 becomes too large relative to drawdowns calculated in Fig. 5.2.) This same method can be extended to calculate a solution for the superposition of any number of pulses with different heights, starting times and durations. A practical application of these results would give a farmer an estimate for the maximum rate and the times during which irrigation water could be spread on a field without submerging the roots of his crop.

	A	B	C	D	E	F	G	H
1	R (m/day)	L (m)	T (m^2/day)	S	nmax	delta (days)	x (m)	
2	0.001	500	2000	0.07	30	0.5	250	
3								
4	t (days)	ABS(s) (m)						
5	0	0						
6	0.05	0.0007145						
7	0.1	0.00142861						
8	0.15	0.00214048						
9								
10								
11								
12								
13								
14								
15								
16								
17								
18								
19								
20								
21								
22								
23								
24								
25	1	0.0055401						
26	1.05	0.00542965						
27	1.1	0.00532411						
28	1.15	0.00522319						
29	1.2	0.00512656						
30	1.25	0.0050339						

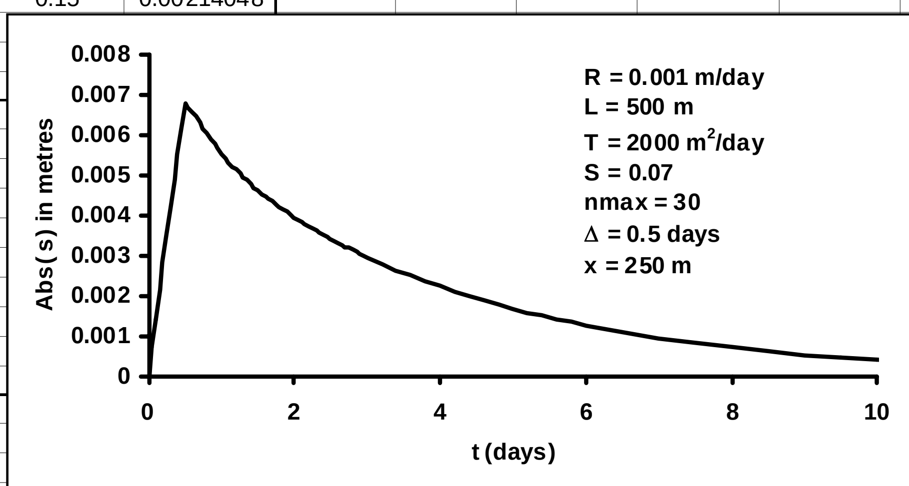


Fig. 5.2. Free surface rise created by a recharge pulse at x = 250 m.

Lecture 6
Spreadsheet Calculations for Combinations of Pulses

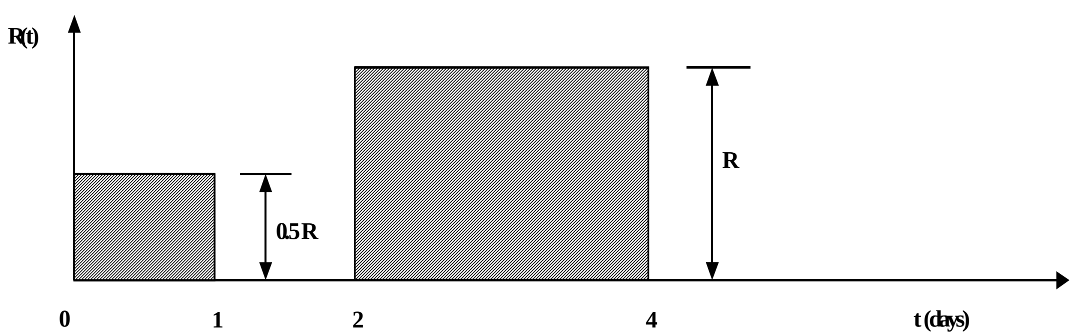
In the previous lecture it was shown that the solution for a pulse of height R that starts at t = 0 and extends to t = Δ is given by

$$s_p(x,t) = s(x,t) - s(x,t-\Delta) \qquad (0 < x < 1,\ -\infty < t < \infty) \qquad (6.1)$$

where $s(x,t)=0$ when $t \le 0$. If we use the time translation principle in Eq.(6.1), then the solution for a pulse of height R that starts at $t = t_k$ and extends to $t = t_k + \Delta_k$ is obtained from Eq.(6.1) by replacing t with $t - t_k$.

$$s_p(x,t-t_k) = s(x,t-t_k) - s[x,t-(t_k+\Delta_k)] \qquad (0 < x < 1,\ -\infty < t < \infty) \quad (6.2)$$

Repeated use of Eq.(6.2) in a spreadsheet allows us to calculate solutions for any combination of pulses. For an example, consider the two-pulse variation for R(t) shown below:



The second pulse has a width of two days and a height R [R is incorporated into the definition of s*, as defined in Eq.(5.2)], while the first pulse has a width of one day and a height of only 0.5 R.

The spreadsheet containing both the calculations and corresponding plot for the two-pulse example is shown in Fig. 6.1. Solutions for pulses one and two are shown in columns B and C, respectively, with the characteristics for each pulse shown in rows 4-7 directly above each solution. The equation

=(B$7*$A$2*$B$2^2/$C$2)*s_pulse($F$2/$B$2,($A10-B$5)*$C$2/($D$2*$B$2^2),
B$6*$C$2/($D$2*$B$2^2),$E$2)

is entered in cell B10, dragged across to cell C10 and then dragged downward to complete the calculations for columns B and C. Finally, the superposition principle is invoked by summing the entries in columns B and C to obtain the solution in column D for the combined two-pulse example. The imbedded plot shows a typical behaviour for this kind of problem, with the free surface rising during periods of recharge and decreasing after recharge stops.

	A	B	C	D	E	F	G	H
1	R (m/day)	L (m)	T (m^2/day)	S	nmax	x (m)		
2	0.001	500	2000	0.07	30	250		
3								
4	Pulse # =	one	two					
5	t_k (days) =	0	2					
6	Δ_k (days) =	1	2					
7	Relative R	0.5	1					
8								
9	t (days)	Pulse Solutions		sum=A+B				
10	0	0	0	0				
11	0.1	0.0007143	0	0.000714				
12	0.2	0.00142286	0	0.001423				
13	0.3	0.00211079	0	0.002111				
14								
15								
16								
17								
18								
19								
20								
21								
22								
23								
24								
25								
26								
27								
28								
29								
30	2	0.00428806	5.77E-18	0.004288				
31	2.1	0.00415684	0.001429	0.005585				
32	2.2	0.00403193	0.002846	0.006878				
33	2.3	0.00391257	0.004222	0.008134				

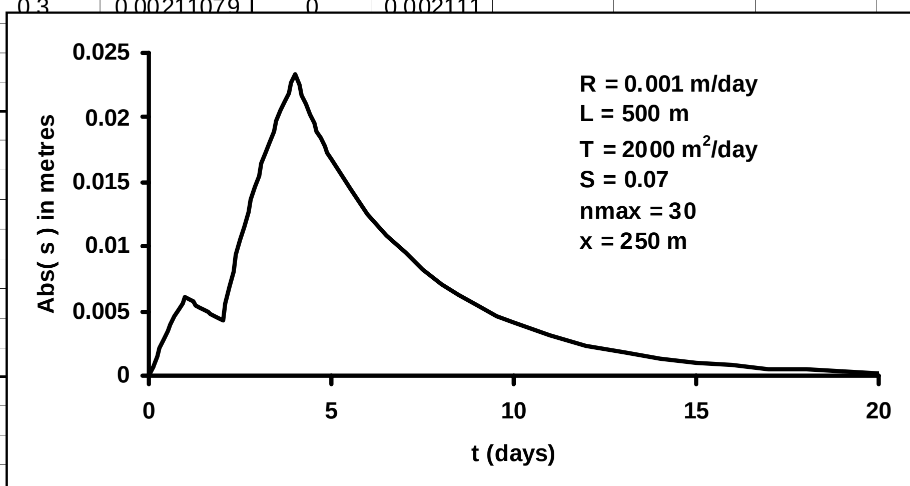
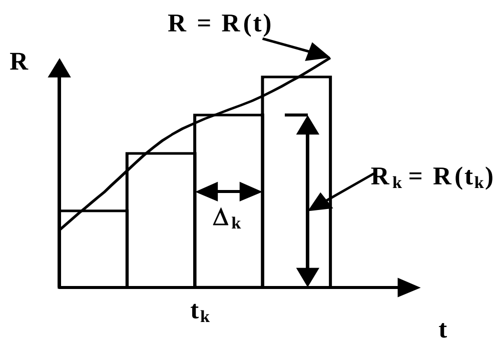


Fig. 6.1. The spreadsheet for a two-pulse example.

It is possible to use the principles just discussed to obtain a well-known result that applies when R is a continuous function of time. This result is known as the Duhamel superposition principle, and it is obtained by approximating R(t) with a distribution of pulses, as shown below.



The principle states that if φ is a function obtained from a linear set of equations with coefficients that do not depend upon t, and if φ is expressible in the form

$$\varphi(x,y,z,t) = R\ f(x,y,z,t) \tag{6.3}$$

where $\varphi = 0$ for $-\infty < t \le 0$ and R = constant, then the solution for φ when R is a continuous function of t is given by any of the following three equations:

$$\varphi = -\int_0^t R(\tau)\,\frac{\partial f(x,y,z,t-\tau)}{\partial \tau}\,d\tau \tag{6.4}$$

$$\varphi = \int_0^t R(\tau)\,\frac{\partial f(x,y,z,t-\tau)}{\partial t}\,d\tau \tag{6.5}$$

$$\varphi = R(0)\ f(x,y,z,t) + \int_0^t R'(\tau)\ f(x,y,z,t-\tau)\,d\tau \tag{6.6}$$

Eq.(6.5) follows from Eq.(6.4) and the identity $\partial f(x,y,z,t-\tau)/\partial t = -\partial f(x,y,z,t-\tau)/\partial \tau$. Eq.(6.6) is obtained by using an integration by parts in Eq.(6.4) and requiring $f(x,y,z,0) = 0$. [$f(x,y,z,0) = 0$ results from Eq.(6.3) and the requirement $\varphi(x,y,z,0) = 0$.] Thus, it is only necessary to show how Eq.(6.4) is obtained.

Since the solution for a pulse of height $R(t_k)$ and width Δ_k at $t = t_k$ is given by $R(t_k)\left\{f(x,y,z,t-t_k) - f\left[x,y,z,t-(t_k+\Delta_k)\right]\right\}$, an approximation for φ when R is a continuous function of t can be obtained by summing the contribution from all pulses under the curve R = R(t).

$$\varphi = \sum_{k=0}^{\infty} R_k \left\{ f\left(x, y, z, t - t_k\right) - f\left[x, y, z, t - \left(t_k + \Delta_k\right)\right] \right\} \tag{6.7}$$

Multiply the right side of Eq.(6.7) by Δ_k / Δ_k and regroup terms to obtain

$$\varphi = -\sum_{k=0}^{\infty} R\left(t_k\right) \frac{f\left[x, y, z, t - (t_k + \Delta_k)\right] - f\left(x, y, z, t - t_k\right)}{\Delta_k} \Delta_k \tag{6.8}$$

Now take the limit $\Delta_k \to 0$ to obtain

$$\varphi = -\int_0^{\infty} R\left(\tau\right) \frac{\partial f\left(x, y, z, t - \tau\right)}{\partial \tau} d\tau \tag{6.9}$$

Finally, since $\varphi\left(x, y, z, t\right) = 0$ for $-\infty < t \le 0$, the identity $\partial f\left(x, y, z, t - \tau\right) / \partial \tau = 0$ holds for $t < \tau < \infty$. Therefore, the upper limit in Eq.(6.9) can be replaced with t, which reduces Eq.(6.9) to Eq.(6.4). Readers wanting to see a different way to arrive at this same result are advised to look in Hildebrand (1976).

The Duhamel superposition principle, as expressed by any of the three forms given by Eqs.(6.4)-(6.6), is encountered in structural dynamics when analysing the behaviour of structures under seismic loadings. The Duhamel integral is a specialized form of what more generally is referred to as a “convolution integral.” A convolution integral, I(t), has the following form:

$$I\left(t\right) = \int_0^{t} f\left(\tau\right) g\left(t - \tau\right) d\tau \tag{6.10}$$

A second equivalent form can be obtained from Eq.(6.10) by making the substitutions $\xi = t - \tau$ and $d\xi = -d\tau$.

$$I\left(t\right) = -\int_t^{0} f\left(t - \xi\right) g\left(\xi\right) d\xi = \int_0^{t} f\left(t - \xi\right) g\left(\xi\right) d\xi \tag{6.11}$$

Then replacing the integration variable ξ with τ in Eq.(6.11) gives the more usual equivalent form of Eq.(6.10).

$$I\left(t\right) = \int_0^{t} f\left(t - \tau\right) g\left(\tau\right) d\tau \tag{6.12}$$

Convolution integrals have many civil engineering applications. For example, they are used in fire engineering, linear systems engineering and risk analysis, and convolution integrals can be used to derive the entire theory of the unit hydrograph in hydrology [Dooge (1973), Hunt, (1985)].

Although Eqs.(6.4)-(6.6) are all mathematically equivalent, it is sometimes preferable to use the form given by Eq.(6.6). This is because $f(x, y, z, t)$ is often given in the form of either an infinite series or a definite integral, and differentiation, as required in the use of either Eq.(6.4) or (6.5), can slow up or completely stop the convergence of an infinite series or definite integral.

In some applications, either f(t) or g(t) in Eq.(6.10) are given as tabulated functions rather than by analytical expressions. It is relatively difficult to transfer arrays between a spreadsheet and a user-defined program, and this means that the numerical evaluation of a convolution integral is not done conveniently with spreadsheet calculations unless the functions f(t) and g(t) are both given by analytical expressions. If this is not the case, then a convolution integral is best evaluated numerically by using a more flexible programming language such as Matlab or Fortran. (Fortran is known to be much more efficient than Matlab for really large problems.) On the other hand, if a convolution integral arises in the context of the Duhamel superposition principle, so that the analytical solution for a pulse is readily calculated, then the spreadsheet computation demonstrated in Fig. 6.1 can be used to evaluate the integral by approximating the continuous function R(t) with a collection of pulses.

Lectures 7 and 8

Taking the Limit $L \to \infty$ in Fig. 2.1

In many applications the flood plain has an unknown but very large width, L. One way to account for this is to evaluate the solution given by Eq.(4.11) for larger and larger values of L until $s(x,t)$ stops changing for the particular values of x and t under consideration. A better way is to rewrite the solution and then take the limit $L \to \infty$. The summation sign becomes a definite integral in this limit, and the resulting solution is identical with the result obtained by solving the problem more directly with an integral known as the "Fourier sine transform." [See Zill and Cullen (1992) for various forms of the Fourier transform.]

The solution given by Eq.(4.11) is

$$s(x,t) = -2\sum_{n=1}^{\infty} \frac{\left(1-e^{-t\alpha_n^2}\right)}{\alpha_n^3} \sin(\alpha_n x) \qquad \left[0 < x < 1,\ 0 < t < \infty,\ \alpha_n = (2n-1)\frac{\pi}{2}\right] \quad (7.1)$$

in which all variables are dimensionless and are defined, in Eq.(2.13), as follows:

$$(s^*, x^*, t^*) = \left(\frac{sT}{RL^2}, \frac{x}{L}, \frac{tT}{SL^2}\right) \quad (7.2)$$

The asterisk superscript has been omitted in Eq.(7.1) for notational convenience.

Since L appears in the definition of the dimensionless variables in Eq.(7.2), we must rewrite Eq.(7.1) in dimensional variables before taking the limit $L \to \infty$. This gives the following result:

$$\frac{sT}{RL^2} = -2\sum_{n=1}^{\infty} \frac{\left(1-e^{-\frac{tT}{SL^2}\alpha_n^2}\right)}{\alpha_n^3} \sin\left(\alpha_n \frac{x}{L}\right) \qquad \left(0 < \frac{x}{L} < 1,\ 0 < \frac{tT}{SL^2} < \infty\right) \quad (7.3)$$

which can be rearranged slightly to obtain

$$s(x,t) = -2\frac{RL^2}{T}\sum_{n=1}^{\infty} \frac{\left(1-e^{-\frac{tT}{SL^2}\alpha_n^2}\right)}{\alpha_n^3} \sin\left(\alpha_n \frac{x}{L}\right) \qquad (0 < x < L,\ 0 < t < \infty) \quad (7.4)$$

Now make the substitution $\xi_n = \alpha_n / L$ in Eq.(7.4).

$$s(x,t) = -2\frac{R}{T}\sum_{n=1}^{\infty}\frac{1-e^{-\xi_n^2 \frac{tT}{S}}}{L\xi_n^3}\sin(\xi_n x) \qquad (0 < x < L,\ 0 < t < \infty) \qquad (7.5)$$

Since $\xi_n = \alpha_n / L$, we see that an increment in ξ_n is given by

$$\Delta\xi_n = \xi_{n+1} - \xi_n = \frac{\alpha_{n+1}}{L} - \frac{\alpha_n}{L} = \frac{\left[2(n+1)-1\right]\pi/2}{L} - \frac{(2n-1)\pi/2}{L} = \frac{\pi}{L} \qquad (7.6)$$

Therefore, setting $1/L = \Delta\xi_n / \pi$ in Eq.(7.5) gives

$$s(x,t) = -\frac{2}{\pi}\frac{R}{T}\sum_{n=1}^{\infty}\frac{1-e^{-\xi_n^2 \frac{tT}{S}}}{\xi_n^3}\sin(\xi_n x)\,\Delta\xi_n \qquad (0 < x < L,\ 0 < t < \infty) \qquad (7.7)$$

Eq.(7.6) shows that $\Delta\xi_n \to 0$ as $L \to \infty$. Therefore, letting $L \to \infty$ in Eq.(7.7) gives the final result

$$s(x,t) = -\frac{2}{\pi}\frac{R}{T}\int_0^{\infty}\frac{1-e^{-\xi^2 \frac{tT}{S}}}{\xi^3}\sin(\xi x)\,d\xi \qquad (0 < x < \infty,\ 0 < t < \infty) \qquad (7.8)$$

Although Eq.(7.8) can be obtained by direct use of the Fourier sine transform, we will not introduce use of the various Fourier transform pairs in this course. **Therefore, you will be expected to be able to use the substitution $\xi_n = \alpha_n / L$ to convert a Fourier series solution for a finite value of L to a Fourier integral solution, such as Eq.(7.8), for an infinite value of L.**

The integrand in Eq.(7.8) is finite at the lower integration limit, $\xi = 0$, and vanishes fast enough as $\xi \to \infty$ to ensure that the integral exists.[5] However, it can be put in a form more convenient for numerical evaluation by rewriting it in dimensionless variables. Since the argument of any transcendental function must be dimensionless, and since x has units of

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[5] In most cases, an integral with an infinite upper limit will exist only if the integrand vanishes at least as fast as $1/\xi^{1+\varepsilon}$, where $\varepsilon > 0$. However, exceptions do occur. For example, $\int_0^{\infty}\frac{\sin(\xi)}{\xi}\,d\xi = \frac{\pi}{2}$ because of cancellations caused by oscillations in $\sin(\xi)$ as $\xi \to \infty$.

length, it becomes obvious that ξ has units of length^{-1}. Therefore, introduce the following dimensionless variables into Eq.(7.8):

$$\left(s^*, t^*, u\right) = \left(\frac{sT}{Rx^2}, \frac{tT}{Sx^2}, \xi x \right) \tag{7.9}$$

Introducing the dimensionless variables defined in Eq.(7.9) into Eq.(7.8) and then omitting the asterisk superscript for notational convenience gives the following simpler result:

$$s(t) = -\frac{2}{\pi}\int_0^\infty \frac{1-e^{-tu^2}}{u^3}\sin(u)\,du \tag{7.10}$$

Eq.(7.8) shows that s is a function of two independent variables, x and t, but Eq.(7.10) shows that s* has been reduced to a function of only one independent variable, t*. A solution with this property is known as a "similarity solution," and the variables s* and t* are called "similarity variables." Many applied problems in engineering, particularly in fluid mechanics and heat transfer, are described by similarity solutions.

The integral on the right side of Eq.(7.10) can be evaluated exactly.[6] However, this is not possible in most cases, and it is probably more important that you learn to evaluate this type of integral numerically. The first step in the process is to inspect the integrand to see if it becomes singular (infinite) at any point on the interval of integration. The only place this might happen in Eq.(7.10) is at the lower limit, $u = 0$. However, the asymptotic behaviours $e^{-tu^2} \sim 1 - tu^2 + \cdots$ and $\sin(u) \sim u + \cdots$ as $u \to 0$ show that

$$\lim_{u\to 0}\frac{1-e^{-tu^2}}{u^3}\sin(u) = t \tag{7.11}$$

[6] Start by differentiating Eq.(7.10) to obtain $\frac{ds(t)}{dt} = -\frac{2}{\pi}\int_0^\infty e^{-tu^2}\,\frac{\sin(u)}{u}\,du = \operatorname{erf}\left(\frac{1}{2\sqrt{t}}\right)$, where erf(x) = error function. Then integration from 0 to t gives $s(t) = \frac{1}{2}\operatorname{erfc}\left(\frac{1}{2\sqrt{t}}\right) - t\,\operatorname{erf}\left(\frac{1}{2\sqrt{t}}\right) - \sqrt{\frac{t}{\pi}}\;e^{-\frac{1}{4t}}$ where erfc(x) = complementary error function.

Therefore, the integrand has a finite value at every point on the integration interval, although we will have to tell the computer to make use of the result in Eq.(7.11) at the lower integration limit. (No computer is bright enough to recognize and evaluate an indeterminate form like 0/0. Doesn't that make you feel superior to a machine!)

A second difficulty occurs with the upper integration limit. It is not possible to integrate numerically to infinity. One possibility is to change integration variables and use one, or several, transformations to map the interval $0 \le u \le \infty$ into a finite interval. In this case, however, the function $\sin(u)$ has an infinite number of oscillations on the infinite interval, and any attempt to transform the infinite interval to a finite interval will compress all of these oscillations into the finite interval. The result is neither pretty nor easy to work with when trying to use a quadrature formula such as the trapezoidal rule or Simpson's rule to evaluate the resulting integral. Therefore, we will divide the interval $0 \le u \le \infty$ into the sum of the intervals $0 \le u \le b$ and $b \le u \le \infty$ and rewrite Eq.(7.10) in the following way:

$$s(t) = -\frac{2}{\pi}\int_0^b \frac{1-e^{-tu^2}}{u^3}\,\sin(u)\,du + \varepsilon \quad \left[\varepsilon = -\frac{2}{\pi}\int_b^\infty \frac{1-e^{-tu^2}}{u^3}\,\sin(u)\,du\right] \tag{7.12}$$

Then we will choose the upper limit, b, so that it is large enough to ensure that the contribution from ε can be neglected.

There are two ways that a rational choice can be made for b in Eq.(7.12). One way is to neglect ε and increase b in Eq.(7.12) until two successive evaluations of s(t) give nearly identical results. This is the way that is used most often in practice. However, the relatively simple form of the integrand in this particular case gives us another possibility. We can make use of the following inequality to estimate an upper bound for ε:

$$|\varepsilon| = \frac{2}{\pi}\int_b^\infty \frac{\left|1-e^{-tu^2}\right|}{\left|u^3\right|}\,\left|\sin(u)\right|\,du \le \frac{2}{\pi}\int_b^\infty \frac{du}{u^3} = \frac{1}{\pi b^2} \tag{7.13}$$

Thus, Eq.(7.13) can be used to ensure that the chosen value for b is large enough to make ε negligible.

Numerical quadrature is probably accomplished most often by using either the trapezoidal rule or Simpson's rule. [See your text from last year by Zill and Cullen (1992) for a discussion of these two quadrature formulae.] The trapezoidal rule approximates an integrand between every two nodes with a straight line, while Simpson's rule approximates an integrand between every three nodes with a second-degree polynomial. (Thus, Simpson's rule requires an even number of "slices.") Since Simpson's rule allows for integrand curvature and also has a smaller truncation error, it is the most accurate choice for a well-behaved integrand. However, examples sometimes occur in which an integrand changes extremely rapidly at one or more points along the integration interval. When this happens, the trapezoidal rule with a small nodal spacing can give better accuracy than either Simpson's rule or some other higher-order quadrature formula.

The following two Visual Basic programs allow you to evaluate an integral with either the trapezoidal rule or Simpson' rule:

```
'This calculates the integral of funct(x) from x=a to  x=b using the trapezoidal
'rule with n slices. funct(x) must be computed with another user-defined program.
Function trapz(a, b, n)
   delta = (b - a) / n
   trapz = 0#
   y_1 = funct(a)
   For i = 1 To n
```

```
      y_2 = funct(a + i * delta)
      trapz = trapz + delta * (y_1 + y_2) / 2
      y_1 = y_2
   Next i
End Function

'This calculates the integral of funct(x) from x=a to  x=b using Simpson's one-third
'rule with n slices. n must be an even number, and funct(x) must be computed with
'another user-defined program.
Function Simp(a, b, n)
   delta = (b - a) / n
   Simp = 0#
   y_1 = funct(a)
   For i = 1 To n - 1 Step 2
      y_2 = funct(a + i * delta)
      y_3 = funct(a + (i + 1) * delta)
      Simp = Simp + delta * (y_1 + 4 * y_2 + y_3) / 3
      y_1 = y_3
   Next i
End Function
```

Either one of these two programs requires another user-defined program to evaluate the integrand. For example, the following program evaluates the integrand in Eq.(7.12):

```
'This evaluates the integrand in Eq.(7.12). The dimensionless variable t* in this
integrand
'is defined as follows:
'             t*=tT/Sx^2
'All input and output variables are dimensionless, but the asterisk superscript is
'omitted for notational convenience.
Function funct(u, t)
Pi = 3.141592654
If u = 0# Then
   funct = 2 * t / Pi
Else
   funct = 2 * (1 - Exp(-t * u ^ 2)) * Sin(u) / (Pi * u ^ 3)
End If
End Function
```

There are three things worth noting about this program: first, the constant $2/\pi$ outside the integral in Eq.(7.12) has been incorporated into the integrand; second, the first part of the If loop evaluates the integrand at the lower limit u = 0; and third, the integrand is a function of two independent variables. Therefore, the program for our chosen quadrature formula must be rewritten to reflect the fact that a number of variables appear in this problem that do not appear in the programs given above. If Simpson's rule is used, then the program becomes

```
'This evaluates the integral in Eq.(7.12) by using Simpson's one-third
'rule with n slices. n must be an even number, and funct(x) is computed with
'another user-defined program. All input and output variables are dimensionless
'and are defined as follows:
'    s*=abs(s)T/RL^2       t*=tT/Sx^2
'The asterisk superscript has been omitted for notational convenience.
Function s_inf(t, b, n)
If t = 0# Then
   s_inf = 0#
Else
   a = 0#
   delta = (b - a) / n
   s_inf = 0#
   y_1 = funct(a, t)
   For i = 1 To n - 1 Step 2
      y_2 = funct(a + i * delta, t)
      y_3 = funct(a + (i + 1) * delta, t)
      s_inf = s_inf + delta * (y_1 + 4 * y_2 + y_3) / 3
      y_1 = y_3
   Next i
End If
End Function
```

It is always a good idea to plot the integrand before attempting to evaluate an integral numerically. This is to make sure that nodes are spaced closely enough to allow the integrand to be approximated with straight-line segments if the trapezoidal rule is used or with segments of a parabola if Simpson's rule is used. Fig. 7.1 shows a plot of the integrand in Eq.(7.12) calculated at 101 equally spaced points along the interval $0 \le u \le 20$. These points are close enough to allow the integrand to be approximated closely with segments of a parabola, but straight-line segments would give a much less accurate approximation with this spacing. Hence, Simpson's rule is a good choice if we choose n = 100 and b = 20 in the above program.

Fig. 7.2 shows a plot of $s(t)$ calculated from Eq.(7.12). Results calculated numerically with Simpson's rule, using n = 100 and b = 20, are plotted with a solid curve, and values calculated from the exact value of Eq.(7.10) are shown with unfilled circles. The comparison is very close, and Eq.(7.13) gives the following upper bound for the error created by neglecting the part of the integral from u = b to $u = \infty$:

$$|\varepsilon| \le \frac{1}{\pi b^2} = \frac{1}{\pi(20)^2} = 0.0008 \tag{7.14}$$

Of course, this does not include an estimate for the error caused by approximating the first integral with Simpson's rule.

	I	J	K	L	M	N	O	P
1	b	n	delta					
2	20	100	0.2					
3								
4	t* =	1	5	10				
5								
6	u	Values of the integrand						
7	0	0.63662	3.183099	6.366198				
8	0.2	0.619903	2.865795	5.212109				
9	0.4	0.572738	2.133088	3.091546				
10	0.6	0.503121	1.389092	1.618707				
11								
12								
13								
14								
15								
16								
17								
18								
19								
20								
21								
22								
23								
24								
25								
26								
27	4	-0.007528	-0.007528	-0.007528				
28	4.2	-0.007489	-0.007489	-0.007489				
29	4.4	-0.007112	-0.007112	-0.007112				
30	4.6	-0.006499	-0.006499	-0.006499				
31	4.8	-0.005734	-0.005734	-0.005734				
32	5	-0.004884	-0.004884	-0.004884				

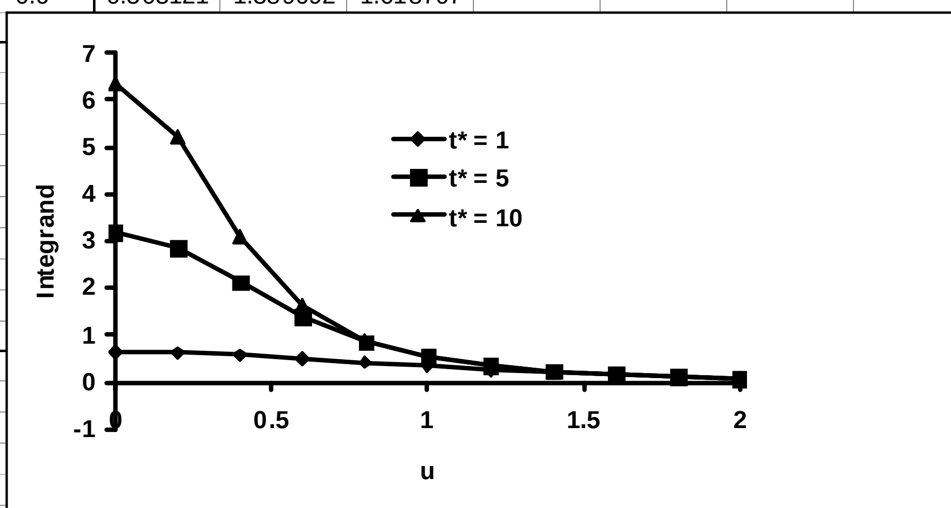


Fig. 7.1. A plot of the integrand in Eq.(7.12)

	A	B	C	D	E	F	G
1	R (m/day)	L (m)	T (m^2/day)	S	b	n	
2	0.001	infinite	2000	0.07	20	100	
3							
4	t*	s*		t*	Exact s*		
5	0	0		0	0		
6	0.1	0.09939463		0.5	0.42466		
7	0.2	0.19255458		1	0.720141		
8	0.3	0.27689281		1.5	0.957503		
9	0.4	0.35375537		2	1.161443		
10	0.5	0.42461825		2.5	1.343009		
11	0.6	0.49061749		3	1.508252		
12	0.7	0.55259821		3.5	1.660911		
13	0.8	0.61119678		4	1.803483		
14							
15							
16							
17							
18							
19							
20							
21							
22							
23							
24							
25							
26							
27							
28							
29							
30	2.5	1.34296663					
31	2.6	1.37717756					
32	2.7	1.41077706					
33	2.8	1.44379673					

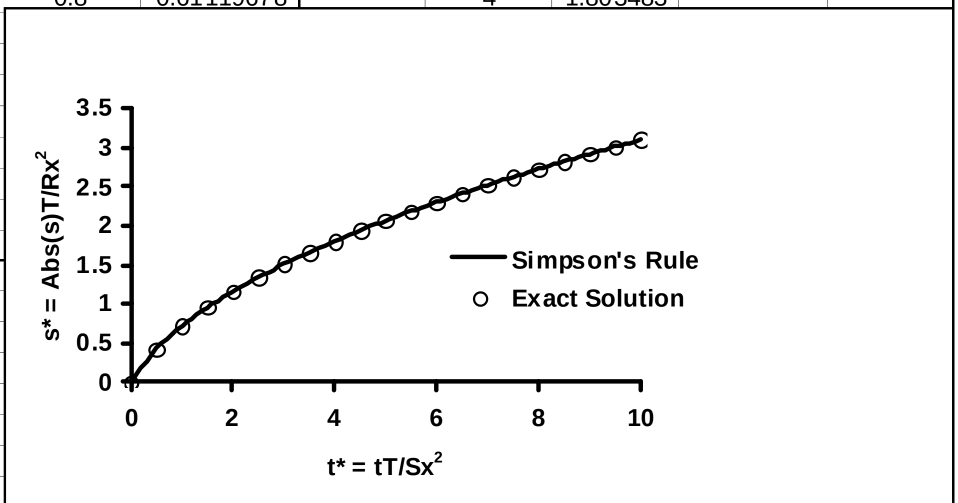


Fig. 7.2. A plot of the solution for s(t).

Lecture 9

An Inhomogeneous Initial Condition

In lecture 4 we calculated the solution for a problem with homogeneous boundary conditions and an inhomogeneous partial differential equation. In this lecture we will show that the same general method can also be applied to a problem with an inhomogeneous initial condition provided that the boundary conditions are homogeneous. In particular, we will set R = 0 and the initial elevation of the free surface equal to a constant, H, for the problem shown in Fig. 2.1. Thus, our dimensional problem statement becomes

$$T\frac{\partial^2 s}{\partial x^2} = S\frac{\partial s}{\partial t} \qquad (0 < x < L,\ 0 < t < \infty) \tag{9.1}$$

$$s(0,t) = 0 \qquad (0 < t < \infty) \tag{9.2}$$

$$\frac{\partial s(L,t)}{\partial x} = 0 \qquad (0 < t < \infty) \tag{9.3}$$

$$s(x,0) = -H \qquad (0 < x < L) \tag{9.4}$$

where -H is a negative constant that indicates a free surface rise. The constants L and H are obvious scales for x and s, respectively, and, if t_0 is our unknown time scale, then terms in Eq.(9.1) have the following scales:

$$T\frac{H}{L^2} \qquad S\frac{H}{t_0} \tag{9.5}$$

Therefore, relative scales in (9.5) become

$$\frac{t_0 T}{SL^2} \qquad 1 \tag{9.6}$$

and we see that coefficients of the dimensionless form of Eq.(9.1) will be unity if we choose $t_0 = SL^2/T$. In this way, we arrive at the following dimensionless variables for use in Eqs.(9.1)-(9.4):

$$(s^*, x^*, t^*) = \left(-\frac{s}{H}, \frac{x}{L}, \frac{tT}{SL^2}\right) \tag{9.7}$$

The minus sign has been introduced, arbitrarily, in Eq.(9.7) so that calculated values of s* will be positive. Introducing the dimensionless variables in Eq.(9.7) into Eqs.(9.1)-(9.4) gives the following problem:

$$\frac{\partial^2 s}{\partial x^2} = \frac{\partial s}{\partial t} \qquad (0 < x < 1,\ 0 < t < \infty) \tag{9.8}$$

$$s(0,t) = 0 \qquad (0 < t < \infty) \tag{9.9}$$

$$\frac{\partial s(1,t)}{\partial x} = 0 \qquad (0 < t < \infty) \tag{9.10}$$

$$s(x,0) = 1 \qquad (0 < x < 1) \tag{9.11}$$

where the asterisk superscript has been omitted for notational convenience.

The form of the homogeneous boundary conditions, Eqs.(9.9)-(9.10), causes us to look for a solution in the form of the following half-range Fourier series:

$$s(x.t) = \sum_{n=1}^{\infty} a_n(t) \sin(\alpha_n x) \qquad (0 < x < 1,\ 0 < t < \infty) \tag{9.12}$$

where the homogeneous boundary condition at x = 0 has determined that we use a sine series and the homogeneous boundary condition at x = 1 determines

$$\alpha_n = (2n-1)\frac{\pi}{2} \tag{9.13}$$

Thus, Eqs.(9.12)-(9.13) satisfy the boundary conditions (9.9)-(9.10), and direct substitution shows that Eq.(9.8) will be satisfied if

$$\sum_{n=1}^{\infty}\left[\frac{da_n(t)}{dt} + \alpha_n^2 a_n(t)\right] \sin(\alpha_n x) = 0 \tag{9.14}$$

Eq.(9.14) will be satisfied if the quantity in brackets vanishes, and the Fourier series orthogonality relationships can be used to show that the quantity in brackets must vanish.[7] This gives the following differential equation for $a_n(t)$:

[7] These orthogonality relationships hold because of the homogenous boundary conditions, (9.9) and (9.10). Review lecture 3 if you have forgotten this.

$$\frac{da_n}{dt} + \alpha_n^2 a_n = 0 \qquad (0 < t < \infty) \tag{9.15}$$

This first-order ordinary differential equation requires one initial condition, which can be found by inserting (9.11) in Eq.(9.12).

$$s(x,0) = \sum_{n=1}^{\infty} a_n(0)\sin(\alpha_n x) = 1 \qquad (0 < x < 1) \tag{9.16}$$

Multiplying Eq.(9.16) by $\sin(\alpha_m x)$, integrating from x = 0 to x = 1 and using the orthogonality relationships gives

$$a_m(0)\,\frac{1}{2} = \int_0^1 \sin(\alpha_m x)\,dx = \frac{1}{\alpha_m} \tag{9.17}$$

in which we have used $\cos(\alpha_m) = 0$. Thus, our initial condition for Eq.(9.15) becomes

$$a_n(0) = \frac{2}{\alpha_n} \tag{9.18}$$

(We can replace the subscript m in Eq.(9.17) with n provided that we do so on both sides of the equation.) The solution of Eqs.(9.15) and (9.18) is given by

$$a_n(t) = 2\frac{e^{-t\alpha_n^2}}{\alpha_n} \tag{9.19}$$

which can be inserted in Eq.(9.12) to obtain the final result.

$$s(x,t) = 2\sum_{n=1}^{\infty} \frac{\sin(\alpha_n x)}{\alpha_n}\, e^{-t\alpha_n^2} \qquad (0 < x < 1,\ 0 < t < \infty) \tag{9.20}$$

Eq.(9.20) is plotted in Fig. 9.1, using dimensionless variables, to ensure that the solution behaviour appears reasonable.

The dimensionless plot in Fig. 9.1 illustrates an extremely important point in regard to the use of scaled dimensionless variables. Scales for each of the variables [s has been scaled with H, x with L and t with SL^2/T] were chosen so that the dimensionless terms in Eq. (9.7) each had an expected maximum magnitude of unity. More specifically, H was the expected maximum magnitude of s, L was the maximum value of x and $t_0 = SL^2/T$ was chosen so that the right and left sides of Eq.(9.1) have the same order of magnitudes. As a result, it can be expected that most of the changes in s with t will occur during the time interval $0 < \frac{tT}{SL^2} < O(1)$, where $O(1)$ is an order of magnitude notation meaning that this term may have a magnitude anywhere between 0.5 to 2 or 3 but not as small as 0.1 or as large as 10. Fig. 9.1 shows that these scaling factors were chosen perfectly since s* and x* both range from zero to one and movement of the free surface almost ceases by the time t* reaches a value of one. Correct scales for problem variables are not always easy to deduce, but this process of scaling variables and terms in an equation is extremely useful in applications since it frequently allows us to estimate beforehand which terms in an equation are small enough to neglect. For example, the German mechanical engineer Ludwig Prandtl used this technique in 1904 to simplify the Navier-Stokes equations of fluid mechanics and obtain a set of equations referred to as the "boundary-layer equations." Prandtl's 1904 paper probably made the single most important contribution of the twentieth century to our fundamental knowledge of fluid mechanics. Furthermore, although Prandtl was a research engineer and not a mathematician, his boundary-layer concept gave birth to a branch of mathematics known as singular perturbation theory, which provides one of the few available means of attacking some of the most difficult non-linear problems in engineering and applied mathematics. [See, for example, the book by Van Dyke (1975).]

We cannot use the Duhamel superposition principle to obtain a solution when H is a function of time since an initial condition cannot be a function of t. (Make sure that you understand why!) However, we can take the limit $L \to \infty$ by rewriting the solution in dimensional variables to obtain

$$\frac{-s(x,t)}{H} = 2\sum_{n=1}^{\infty} \frac{\sin\left(\alpha_n \frac{x}{L}\right)}{\alpha_n} e^{-\frac{tT}{S}\left(\frac{\alpha_n}{L}\right)^2} \tag{9.21}$$

Then make the substitutions $\xi_n = \alpha_n / L$ and $\Delta\xi_n = \xi_{n+1} - \xi_n = \pi / L$ in Eq.(9.21).

$$\frac{-s(x,t)}{H} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(\xi_n x)}{\xi_n}\ e^{-\frac{tT}{S}\xi_n^2}\ \Delta\xi_n \tag{9.22}$$

Therefore, taking the limit $\Delta\xi_n \to 0$ as $L \to \infty$ gives the result for a semi-infinite aquifer.

$$\frac{-s(x,t)}{H} = \frac{2}{\pi}\int_0^{\infty}\frac{\sin(\xi x)}{\xi}\ e^{-\frac{tT}{S}\xi^2}\ d\xi = \operatorname{erf}\left(\frac{x}{2}\sqrt{\frac{S}{tT}}\right) \tag{9.23}$$

where $\operatorname{erf}(x) =$ error function. The solution given by Eq.(9.23) is a similarity solution that can be written using only two dimensionless variables:

$$s^* = \operatorname{erf}\left(\frac{x^*}{2}\right) \tag{9.24}$$

where

$$(s^*, x^*) = \left(\frac{-s}{H}, x\sqrt{\frac{S}{tT}}\right) \tag{9.25}$$

	A	B	C	D	E	F	G	H
1	nmax							
2	30							
3								
4	t =	0	0.001	0.01	0.1	0.5	1	
5								
6	x							
7	0	1	0	0	0	0	0	
8	0.02	1	0.345275	0.112463	0.035667	0.0116472	0.003392	
9	0.04	1	0.62891	0.222703	0.071263	0.02328291	0.00678	
10	0.06	1	0.820289	0.328627	0.106717	0.03489562	0.010162	
11								
12								
13								
14								
15								
16								
17								
18								
19								
20								
21								
22								
23								
24								
25								
26								
27	0.4	1	0.999998	0.995322	0.62856	0.21794734	0.063467	
28	0.42	1	1	0.997021	0.651935	0.22726189	0.06618	
29	0.44	1	1.000002	0.998137	0.674334	0.23635211	0.068827	
30	0.46	1	0.999999	0.998857	0.695754	0.24520903	0.071407	
31	0.48	1	0.999999	0.999311	0.716192	0.25382392	0.073915	

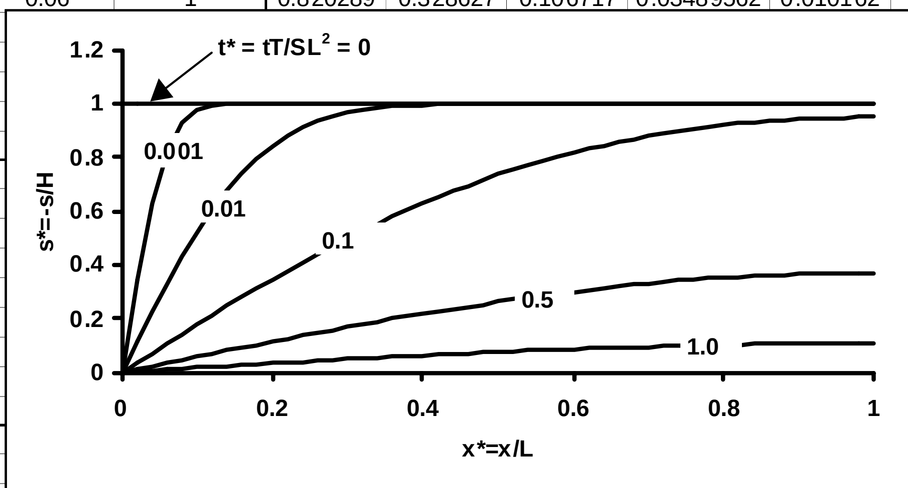


Fig. 9.1. A spreadsheet plot of the solution given by Eq.(9.20).

The error function, $\operatorname{erf}(x)$, and the complementary error function, $\operatorname{erfc}(x)$, are encountered frequently in solutions of the unsteady heat conduction equation. They are defined by the following definite integrals:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2}\, d\xi \qquad (-\infty < x < \infty) \tag{9.26}$$

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\xi^2}\, d\xi \qquad (-\infty < x < \infty) \tag{9.27}$$

where the normalizing constant $2/\sqrt{\pi}$ has been chosen so that $\operatorname{erf}(\infty) = \operatorname{erfc}(0) = 1$. They also satisfy the important identity $\operatorname{erf}(x) + \operatorname{erfc}(x) = 1$. These two integrals cannot be evaluated in closed form but, usually, are evaluated with the use of infinite series. Abramowitz and Stegun (1970) contain an extensive tabulation of their properties. You can use a spreadsheet to calculate their values, but only for $0 < x < \infty$, by using the calls "$= \mathrm{erf}(x)$" and "$= \mathrm{erfc}(x)$." Fig. 9.2 shows a plot of the similarity solution calculated from Eq.(9.24) with the use of these spreadsheet functions. You can begin to obtain an appreciation for the generality of this dimensionless similarity plot by considering the two limits $t \to 0$ and $t \to \infty$ while keeping x fixed.

	A	B	C	D	E	F	G	H
1	x*	s*						
2	0	0						
3	0.05	0.05637198						
4	0.1	0.11246292						
5	0.15	0.16799597						
6	0.2	0.22270259						
7	0.25	0.27632639						
8	0.3	0.32862676						
9								
10								
11								
12								
13								
14								
15								
16								
17								
18								
19								
20								
21								
22								
23								
24								
25	1.15	0.89612382						
26	1.2	0.91031396						
27	1.25	0.92290011						
28	1.3	0.93400793						
29	1.35	0.94376219						
30	1.4	0.95228511						
31	1.45	0.95969502						

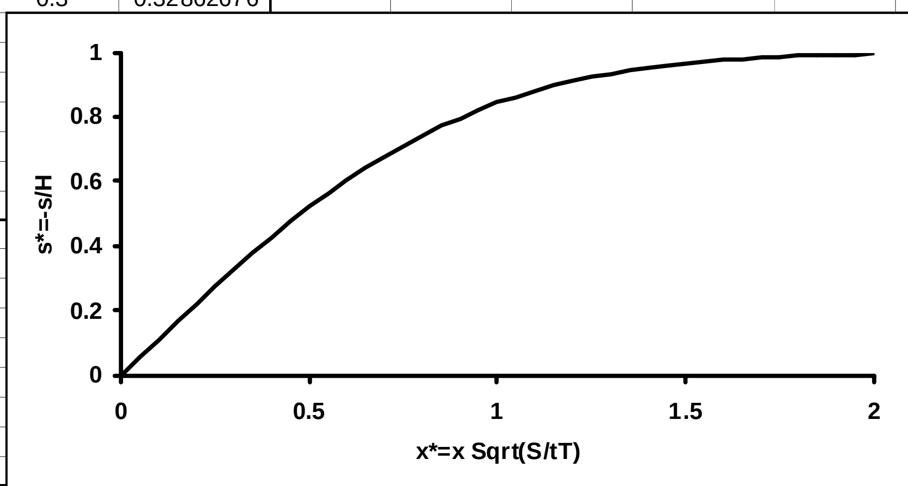


Fig. 9.2. A spreadsheet plot of the similarity solution given by Eq.(9.24).

Hildebrand (1976) derives the following two integrals, which are frequently useful when dealing with solutions of the unsteady heat conduction equation:

$$\int_0^\infty e^{-a^2x^2} \cos(bx)\, dx = \frac{\sqrt{\pi}}{2a}\, e^{-\frac{b^2}{4a^2}} \qquad (a > 0) \tag{9.28}$$

$$\int_0^\infty \frac{\sin(bx)}{x}\, e^{-a^2x^2}\, dx = \frac{\pi}{2}\, \mathrm{erf}\left(\frac{b}{2a}\right) \tag{9.29}$$

Eq.(9.29) was used to evaluate both an integral in footnote 3 of lectures 7-8 and the integral that appears in Eq.(9.23).

Lecture 10

Inhomogeneous Boundary Conditions

In this lecture we will calculate a solution for the problem shown in Fig. 2.1 when the river suddenly rises at t = 0 by an amount H and is held at this constant elevation for an infinite time thereafter. We will set R = 0, and, just to provide a little variation, we will replace the clay embankment at x = L with a reservoir that has zero drawdown for all time. The dimensional problem statement follows:

$$T\frac{\partial^2 s}{\partial x^2} = S\frac{\partial s}{\partial t} \qquad \left(0 < x < L,\ 0 < t < \infty\right) \tag{10.1}$$

$$s\left(0,t\right) = -H \qquad \left(0 < t < \infty\right) \tag{10.2}$$

$$s\left(L,t\right) = 0 \qquad \left(0 < t < \infty\right) \tag{10.3}$$

$$s\left(x,0\right) = 0 \qquad \left(0 < x < L\right) \tag{10.4}$$

It is not difficult to solve this problem when H varies with time. However, the solution when H is constant can be used with the Duhamel superposition principle to calculate the solution when H varies with t. Furthermore, the solution for constant H can be used to calculate the solution for a pulse, and the superposition of translated pulse solutions is a convenient way to obtain numerical approximations with a spreadsheet when H is a specified function of time.

We know from lecture 9 that this problem can be simplified by introducing the following dimensionless variables:

$$\left(s^*, x^*, t^*\right) = \left(\frac{-s}{H}, \frac{x}{L}, \frac{tT}{SL^2}\right) \tag{10.5}$$

Eq.(10.5) transforms Eqs.(10.1)-(10.4) into the following set of dimensionless equations:

$$\frac{\partial^2 s}{\partial x^2} = \frac{\partial s}{\partial t} \qquad \left(0 < x < 1,\ 0 < t < \infty\right) \tag{10.6}$$

$$s\left(0,t\right) = 1 \qquad \left(0 < t < \infty\right) \tag{10.7}$$

$$s\left(1,t\right) = 0 \qquad \left(0 < t < \infty\right) \tag{10.8}$$

$$s\left(x,0\right) = 0 \qquad \left(0 < x < 1\right) \tag{10.9}$$

where the asterisk superscript has been omitted for notational convenience.

The significant difference between the problem considered in this lecture and problems considered in previous lectures is that one of the two boundary conditions is inhomogeneous. Hildebrand (1976) shows how the solution for a set of equations with inhomogeneous boundary conditions can be obtained by inserting a Fourier series directly into the governing equations. However, a solution obtained in this way often has discontinuities at either or both boundaries, and convergence of a Fourier series is slowed considerably near a discontinuity. [See Zill and Cullen (1992) for an example and discussion of "Gibbs phenomenon" and the resulting oscillations that persist even when a large number of terms are used in a series that is used to represent a discontinuous function.] **Therefore, it is better to reformulate any problem with inhomogeneous boundary conditions into another problem that has homogeneous boundary conditions.** This can always be done by defining a new dependent variable, $\varphi(x,t)$, with the following equation:

$$s(x,t) = a + bx + \varphi(x,t) \tag{10.10}$$

where a and b are chosen so that boundary conditions for $\varphi(x,t)$ are homogeneous. [Note: If boundary conditions for $s(x,t)$ are functions of t, then a and b will also be functions of t. If, as in this example, boundary conditions for $s(x,t)$ are constant, then a and b are constants.]

Inserting boundary conditions (10.7) and (10.8) into Eq.(10.10) gives two equations for a and b.

$$s(0,t) = a + 0 + \varphi(0,t) = 1 \tag{10.11}$$

$$s(1,t) = a + b + \varphi(1,t) = 0 \tag{10.12}$$

Since φ must satisfy homogeneous boundary conditions, choose $\varphi(0,t) = 0$ and $\varphi(1,t) = 0$ and solve for a and b to obtain a = 1 and b = -1. Thus, Eq.(10.10) becomes

$$s(x,t) = 1 - x + \varphi(x,t) \tag{10.13}$$

Now Eq.(10.13) can be inserted into Eqs.(10.6)-(10.9) to obtain a problem with homogeneous boundary conditions for $\varphi(x,t)$.

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial \varphi}{\partial t} \qquad (0 < x < 1,\ 0 < t < \infty) \tag{10.14}$$

$$\varphi(0,t) = 0 \qquad (0 < t < \infty) \tag{10.15}$$

$$\varphi(1,t) = 0 \qquad (0 < t < \infty) \tag{10.16}$$

$$\varphi(x,0) = x - 1 \qquad (0 < x < 1) \tag{10.17}$$

As expected, the boundary conditions for φ are homogeneous. However, the initial condition has become inhomogeneous. If either a or b is a function of t, then the partial differential equation for φ also becomes inhomogeneous with an additive term that contains the time derivative of a or b. However, we have already shown that any problem with an inhomogeneous initial condition or partial differential equation can be solved easily and efficiently with a Fourier series **provided that both boundary conditions are homogeneous.** Therefore, whenever you encounter a problem with inhomogeneous boundary conditions, use Eq.(10.10) with appropriately chosen values for a and b to obtain a problem for φ that has homogeneous boundary conditions.

Boundary condition (10.15) determines that a solution for φ must be sought in the form of a sine series.

$$\varphi(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin(\alpha_n x) \qquad (0 < x < 1,\ 0 < t < \infty) \tag{10.18}$$

Boundary condition (10.16) fixes the value for α_n.

$$\alpha_n = n\pi \tag{10.19}$$

Thus, Eqs.(10.18) and (10.19) satisfy both (10.15) and (10.16).

A differential equation for $a_n(t)$ is found by substituting Eq.(10.18) into Eq.(10.14) to obtain the requirement

$$\sum_{n=1}^{\infty} \left[\frac{da_n(t)}{dt} + \alpha_n^2 a_n(t) \right] \sin[\alpha_n x] = 0 \tag{10.20}$$

This equation will be satisfied if we choose $a_n(t)$ so that it is a solution of the following differential equation:

$$\frac{da_n(t)}{dt} + \alpha_n^2 a_n(t) = 0 \qquad (0 < t < \infty) \tag{10.21}$$

This first-order ordinary differential equation requires one initial condition, which can be found by substituting Eq.(10.18) into the initial condition (10.17) to obtain

$$\varphi(x,0) = \sum_{n=1}^{\infty} a_n(0)\,\sin(\alpha_n x) = x - 1 \qquad (0 < x < 1) \tag{10.22}$$

Multipying Eq.(10.22) by $\sin(\alpha_m x)$, integrating from x = 0 to x = 1 and using the Fourier series orthogonality conditions gives

$$a_m(0)\,\frac{1}{2} = \int_0^1 (x-1)\,\sin(\alpha_m x)\,dx = -\frac{1}{\alpha_m} \tag{10.23}$$

where the integral has been evaluated with an integration by parts. This gives the following initial condition:

$$a_n(0) = -\frac{2}{\alpha_n} \tag{10.24}$$

The solution of Eqs.(10.21) and (10.24) is given by

$$a_n(t) = -\frac{2}{\alpha_n}\,e^{-t\alpha_n^2} \tag{10.25}$$

which can be substituted into Eqs.(10.18) and (10.13) to obtain the following solution for the original problem:

$$s(x,t) = 1 - x - 2\sum_{n=1}^{\infty} \frac{\sin(\alpha_n x)}{\alpha_n}\,e^{-t\alpha_n^2} \qquad (0 < x < 1,\ 0 < t < \infty) \tag{10.26}$$

Remember that all variables in Eq.(10.26) are dimensionless with unwritten asterisk superscripts and that these dimensionless variables are defined in Eq.(10.5).

The solution given by Eq.(10.26) can be used to obtain a number of additional solutions. The solution for a semi-infinite aquifer ($L = \infty$) can be obtained by rewriting the equation in dimensional variables.

$$-\frac{s(x,t)}{H} = 1 - \frac{x}{L} - 2\sum_{n=1}^{\infty} \frac{\sin(\alpha_n x/L)}{\alpha_n}\,e^{-\frac{tT}{SL^2}\alpha_n^2} \tag{10.27}$$

Next, set $\xi_n = \alpha_n / L$ and $\Delta\xi_n = \xi_{n+1} - \xi_n = (n+1)\pi/L - n\pi/L = \pi/L$:

$$-\frac{s(x,t)}{H}=1-\frac{x}{L}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(\xi_n x)}{\xi_n}\,e^{-\frac{tT}{S}\xi_n^2}\Delta\xi_n \tag{10.28}$$

Finally, take the limit $L\to\infty$ and $\Delta\xi_n\to 0$ to obtain the solution for a semi-infinfinite aquifer:

$$-\frac{s(x,t)}{H}=1-\frac{2}{\pi}\int_0^{\infty}\frac{\sin(\xi x)}{\xi}\,e^{-\frac{tT}{S}\xi^2}d\xi=1-\operatorname{erf}\left(\frac{x}{2}\sqrt{\frac{S}{tT}}\right)\equiv\operatorname{erfc}\left(\frac{x}{2}\sqrt{\frac{S}{tT}}\right) \tag{10.29}$$

where use has been made of Eq.(9.29) to evaluate the integral. Eq.(10.29) is a similarity solution since it allows $s^*\equiv -s/H$ to be plotted as a function of one independent variable, $x\sqrt{S/(tT)}$, which contains both x and t.

Since Eqs.(10.27) and (10.28) are in the same form as Eq.6.3, and since we can define $s(x,t)=0$ for $t\le 0$ [since $s(x,0)=0$], either of these equations can be inserted in Eq.(6.4), (6.5) or (6.6) to obtain the following solutions when H = H(t):

$$-s(x,t)=-\int_0^t H(\tau)\,\frac{\partial f(x,y,z,t-\tau)}{\partial\tau}\,d\tau \tag{10.30}$$

$$-s(x,t)=\int_0^t H(\tau)\,\frac{\partial f(x,y,z,t-\tau)}{\partial t}\,d\tau \tag{10.31}$$

$$-s(x,t)=H(0)\,f(x,y,z,t)+\int_0^t H'(\tau)\,f(x,y,z,t-\tau)\,d\tau \tag{10.32}$$

where $f(x,y,z,t)$ is the solution for $-s(x,t)$ when H = 1 and, therefore, is given by the right sides of either Eq.(10.27) or Eq.(10.29). Eq.(10.32) is probably the preferable form of the Duhamel integral for numerical calculations since differentiating either an infinite series or a definite integral invariably slows the rate of convergence.

If a Visual Basic program is written that evaluates s* from either Eq.(10.26) or Eq.(10.29) when $t>0$, and if $s^*(x^*,t^*)$ is set equal to zero when $t^*\le 0$, then the solution for a pulse, $s_p(x,t)$, of width Δ and height H can be calculated by setting

$$s_p(x,t)=s(x,t)-s(x,t-\Delta)\qquad(-\infty<t<\infty) \tag{10.33}$$

Fig. 10.1 shows the result of using Eq.(10.33) to make a spreadsheet plot for a pulse width of one day when $L = \infty$. Since the function erfc(x) is unavailable for use in a Visual Basic program, drawdowns in column B were calculated by entering the following IF worksheet function in cell B5:

```
=IF(A5<=0,0,$A$2*ERFC(0.5*$B$2*SQRT($D$2/($C$2*A5))))-
IF(A5<=$E$2,0,$A$2*ERFC(0.5*$B$2*SQRT($D$2/($C$2*(A5-$E$2)))))
```

	A	B	C	D	E	F	G	H
1	H (m)	x (m)	T (m^2/day)	S	Delta (days)			
2	0.5	50	2000	0.07	1			
3								
4	t (days)	Abs(s) (m)						
5	0	0						
6	0.1	0.25416579						
7	0.2	0.31999701						
8	0.3	0.3512747						
9								
10								
11								
12								
13								
14								
15								
16								
17								
18								
19								
20								
21								
22								
23								
24								
25								
26	2.1	0.02165415						
27	2.2	0.01964141						
28	2.3	0.01792956						
29	2.4	0.01645794						
30	2.5	0.01518103						
31	2.6	0.01406399						
32	2.7	0.01307971						

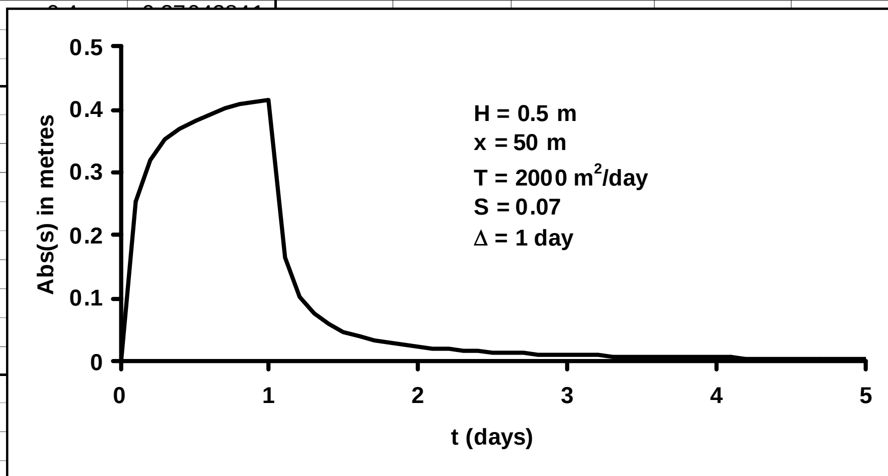
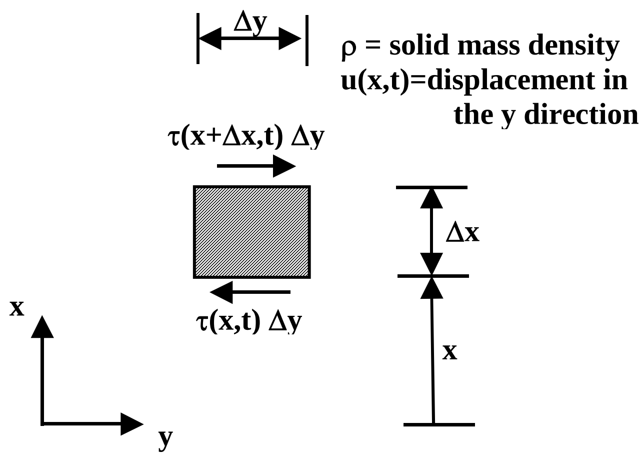


Fig. 10.1. A spreadsheet plot for a pulse calculated from Eqs.(10.33) and (10.29).

Lecture 11

The Wave Equation

The first ten lectures were largely concerned with the unsteady heat conduction equation. In the next few lectures we will consider a second type of equation, known as the wave equation. The wave equation appears in numerous areas of interest to civil engineers.



The above sketch shows an element with unit thickness in an elastic solid subjected to a shear stress distribution, $\tau(x,t)$, created by a transverse displacement distribution, $u(x,t)$. Normal stresses are constant throughout the solid and, therefore, create a zero net force on the element in any direction. Therefore, Newton's second law, which states that the net force on the element equals its mass multiplied by its acceleration, takes the following form:

$$\left[\tau(x+\Delta x,t)-\tau(x,t)\right]\Delta y=(\rho\ \Delta y\ \Delta x)\ \frac{\partial^2 u}{\partial t^2} \tag{11.1}$$

Dividing Eq.(11.1) by the element volume, $(\Delta y\ \Delta x)$, gives

$$\frac{\tau(x+\Delta x,t)-\tau(x,t)}{\Delta x}=\rho\frac{\partial^2 u}{\partial t^2} \tag{11.2}$$

Then taking the limit $\Delta x \to 0$ leads to the result

$$\frac{\partial \tau}{\partial x}=\rho\frac{\partial^2 u}{\partial t^2} \tag{11.3}$$

Hooke's law, which relates a stress to an elastic deformation, takes the following form for this problem:

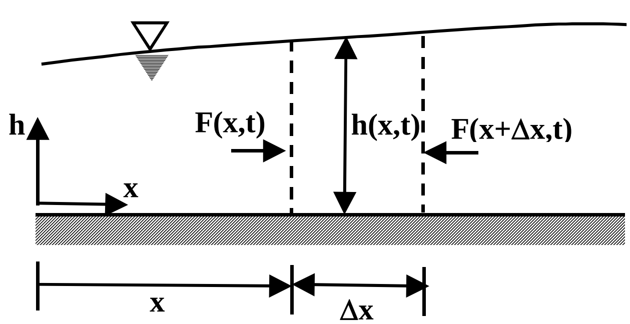
$$\tau = G\frac{\partial u}{\partial x} \tag{11.4}$$

where G is the shear modulus. Finally, eliminating τ from Eqs.(11.3) and (11.4) gives the wave equation.

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad \left(c = \sqrt{G/\rho}\right) \tag{11.5}$$

Eq.(11.5) is useful in civil engineering for calculating the vertical movement of seismic shear waves through the earth or through tall buildings [Clough and Penzien (1975), Humar (1990), Timoshenko and Young (1965)]. We will show in the next lecture that c is the speed (celerity) of an elastic wave (disturbance) in the solid. Typically, in steel we find that $c \approx 5000\ m/s$; in geomechanics, $150\ m/s \le c \le 1000\ m/s$ with the smaller value for a soft soil layer and the larger value for rock; and in a tall building modelled as a shear beam, $c \approx 40\ m/s$.

ρ = fluid mass density



The above sketch shows a gravity wave moving through a fluid that would be at rest if a wave were not present. A system-volume element has a unit thickness and is bounded laterally by the two vertical dashed lines, on the bottom by the reservoir boundary and on top by a free surface. The function $F(x,t)$ is the pressure force on a vertical line extending from the bottom reservoir boundary to the free surface, $h(x,t)$ is the depth of fluid and ρ is the fluid mass density. Newton's second law written for this element has the following form:

$$F(x,t) - F(x+\Delta x, t) = \rho(h\ \Delta x)\left(u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\right) \tag{11.6}$$

where the element acceleration, which appears in brackets on the right side of Eq.(11.6), has been written in Eulerian coordinates.[8] Dividing Eq.(11.6) by Δx and taking the limit $\Delta x \to 0$ gives the following result:

$$-\frac{\partial F}{\partial x} = \rho h\left(u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\right) \tag{11.7}$$

The "long-wave" or "shallow-water" wave approximation[9] assumes that long waves have a small enough free surface curvature to allow pressures to be approximated with a hydrostatic pressure distribution. Thus, the pressure force, $F(x,t)$, on a vertical line is approximated with

$$F(x,t) = \frac{1}{2}\rho g h^2 \tag{11.8}$$

where g = acceleration of gravity. Eliminating F between Eqs.(11.7) and (11.8) gives the non-linear form of Newton's second law for shallow-water waves.

$$-g\frac{\partial h}{\partial x} = u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} \tag{11.9}$$

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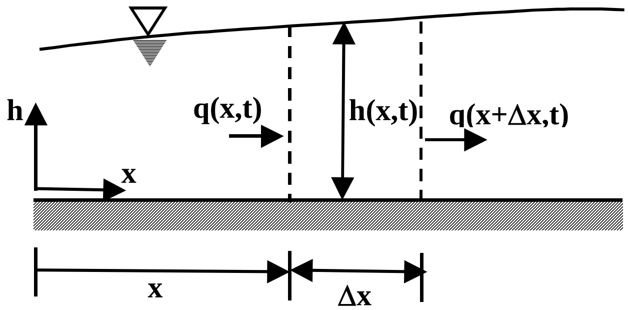
[8] This is one of the important ways that the subjects of fluid mechanics and solid mechanics differ. The acceleration in Eq.(11.1) has been written in Lagrangian coordinates, while the acceleration in Eq.(11.6) has been written in Eulerian coordinates. However, the distinction between Lagrangian and Eulerian coordinates disappears in an elastic solid when the displacements, u(x,t), are relatively small.

[9] "Shallow water" is a relative term that means the water depth is small relative to the wave length. For example, the average ocean depth is about 5 kms, but the wave length for a tsunami is hundreds of kms and the wave length for a tide is about 10,000 kms. Therefore, tsunamis and tides are both shallow-water waves.



Eq.(11.9) contains two unknowns: h and u. A second equation with these two unknowns is obtained from a form of the continuity equation for incompressible flow. The system-volume element in the previous sketch now becomes a control-volume element[10] in the above sketch. The continuity equation for incompressible flow states that the net rate at which fluid passes into the control volume through its lateral boundaries equals the rate at which water is stored within the control volume. If $q(x,t)$ is the volumetric flow through a vertical line extending from the bottom reservoir boundary to the free surface, then the continuity principle becomes

$$q(x,t) - q(x+\Delta x, t) = \frac{\partial(h\ \Delta x)}{\partial t} = \Delta x\ \frac{\partial h}{\partial t} \tag{11.10}$$

Dividing Eq.(11.10) by Δx and taking the limit $\Delta x \to 0$ gives

$$-\frac{\partial q}{\partial x} = \frac{\partial h}{\partial t} \tag{11.11}$$

[10] A system volume has a closed imaginary boundary that deforms with time in a way that keeps the same fluid particles within its boundary. A control volume has a fixed imaginary boundary, which means that fluid passes freely through a control volume boundary.

Finally, since one of the consequences of the long wave approximation is that the velocity distribution along a vertical line is uniform, the flow q is given by $q = uh$ and Eq.(11.11) takes the form

$$\frac{\partial(uh)}{\partial x} + \frac{\partial h}{\partial t} = 0 \qquad (11.12)$$

Eq.(11.12) gives the non-linear form of the continuity equation for shallow-water waves.

In general, Eqs.(11.9) and (11.12) must be solved simultaneously. However, if the waves have a height or amplitude that is small relative to h, a further simplification can be made by setting $h = h_0 + h'$ and $u = u'$, where h_0 is the constant depth that would occur if no waves were present and the primed variables are changes or perturbations from a state of rest caused by the waves. If these expressions for h and u are substituted into Eqs.(11.9) and (11.12), and if products of the primed variables are considered relatively small and, therefore, negligible, Eqs.(11.9) and (11.12) reduce to the following equations:

$$-g\frac{\partial h'}{\partial x} = \frac{\partial u'}{\partial t} \qquad (11.13)$$

$$h_0\frac{\partial u'}{\partial x} + \frac{\partial h'}{\partial t} = 0 \qquad (11.14)$$

Eqs.(11.13) and (11.14) can be used to show that h' and u' are both solutions of the wave equation. For example, multiplying Eq.(11.13) by the constant h_0 and differentiating with respect to x gives

$$-gh_0\frac{\partial^2 h'}{\partial x^2} = \frac{\partial}{\partial x}\left(h_0\frac{\partial u'}{\partial t}\right) = h_0\frac{\partial^2 u'}{\partial x \partial t} \equiv h_0\frac{\partial^2 u'}{\partial t \partial x} = \frac{\partial}{\partial t}\left(h_0\frac{\partial u'}{\partial x}\right) \qquad (11.15)$$

Then use of Eq.(11.14) to eliminate u' from the right side of (11.15) gives

$$-gh_0\frac{\partial^2 h'}{\partial x^2} = -\frac{\partial^2 h'}{\partial t^2} \qquad (11.16)$$

which can be rewritten as

$$c^2\frac{\partial^2 h'}{\partial x^2} = \frac{\partial^2 h'}{\partial t^2} \qquad \left(c = \sqrt{gh_0}\right) \qquad (11.17)$$

A similar manipulation can be used to eliminate h' from Eqs.(11.13) and (11.14) to obtain

$$c^2 \frac{\partial^2 u'}{\partial x^2} = \frac{\partial^2 u'}{\partial t^2} \qquad \left(c = \sqrt{gh_0}\right) \tag{11.18}$$

It is interesting to notice that values for c in Eqs.(11.17) and (11.18) depend only upon the constant g and the undisturbed depth of the reservoir of water. For example, in the open ocean where the average depth is 5 kms, the corresponding wave celerity is $c = 222\ \mathrm{m/s} = 797\ \mathrm{km/hr}$.

The difference in behaviour for solutions of the heat conduction equation

$$k \frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial \varphi}{\partial t} \tag{11.19}$$

and the wave equation

$$c^2 \frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial t^2} \tag{11.20}$$

can be seen by looking for solutions in the form

$$\varphi = \theta(t) \sin(\alpha x) \tag{11.21}$$

This leads to the ordinary differential equations

$$\frac{d\theta(t)}{dt} + k\alpha^2 \theta(t) = 0 \tag{11.22}$$

and

$$\frac{d^2\theta(t)}{dt^2} + c^2\alpha^2 \theta(t) = 0 \tag{11.23}$$

for Eqs.(11.19) and (11.20), respectively. Solution of Eq.(11.22) shows that a solution of the heat conduction equation is given by

$$\varphi(x,t) = A\ e^{-tk\alpha^2}\ \sin(\alpha x) \tag{11.24}$$

which indicates that the sinusoidal variation of $\varphi(x,t)$ at t = 0 is damped exponentially as time increases without creating further oscillations. Contrast this with the solution given by Eqs.(11.23) and (11.21) for the wave equation

$$\varphi(x,t) = \left[A\ \cos(\alpha c t) + B\ \sin(\alpha c t)\right] \sin(\alpha x) \tag{11.25}$$

which indicates that the same sinusoidal variation of $\varphi(x,t)$ at t = 0 oscillates for an infinite time thereafter.

The oscillating nature of solutions of the wave equation can be damped with time by introducing a frictional resistance force in its derivation. If this force is assumed linearly proportional to the first power of the velocity, the wave equation remains linear and has the form

$$c^2 \frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial t^2} + 2\varepsilon \frac{\partial \varphi}{\partial t} \qquad (11.26)$$

where ε is called a damping coefficient. Substituting Eq.(11.21) into Eq.(11.26) shows that the additional damping term leads to the following solution:

$$\varphi(x,t) = e^{-\varepsilon t}\left[A\ \cos\left(t\sqrt{(\alpha c)^2 - \varepsilon^2}\right) + B\ \sin\left(t\sqrt{(\alpha c)^2 - \varepsilon^2}\right)\right]\sin(\alpha x) \qquad (11.27)$$

Thus, sinusoidal oscillations that exist at t = 0 are damped exponentially with time if $\alpha c > \varepsilon$. On the other hand, if $\alpha c < \varepsilon$, solutions of Eq.(11.26) are given by

$$\varphi(x,t) = \left[A\ e^{-t\left(\varepsilon + \sqrt{\varepsilon^2 - (\alpha c)^2}\right)} + B\ e^{-t\left(\varepsilon - \sqrt{\varepsilon^2 - (\alpha c)^2}\right)}\right]\sin(\alpha x) \qquad (11.28)$$

which indicates a behaviour similar to the strongly damped behaviour of heat conduction solutions.[11] This illustrates why the introduction of damping in a structure is an effective way to mitigate damage from an earthquake.

[11] You might find it interesting to compare the discussion in this paragraph with the discussion given in pages 180-187 of Zill and Cullen (1992) concerning the behaviour of solutions for the ordinary differential equation that describes damped motion of a mass suspended from a spring. It is no accident that structural engineers sometimes model seismic induced oscillations of buildings with the "spring equation."

Lecture 12

The General Solution of the Wave Equation

The wave equation without a damping term, Eq.(11.20), is one of only a very small number of second-order partial differential equations for which the general solution is known. In particular, the general solution of

$$c^2 \frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial t^2} \tag{12.1}$$

can be found by setting

$$\varphi(x,t) = f(\xi) \qquad (\xi = x + at) \tag{12.2}$$

where a = constant. The chain rule allows us to calculate the following partial derivatives of φ:

$$\frac{\partial \varphi}{\partial x} = \frac{df(\xi)}{d\xi}\frac{\partial \xi}{\partial x} = \frac{df(\xi)}{d\xi} \tag{12.3}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{df(\xi)}{d\xi}\right] = \frac{d}{d\xi}\left[\frac{df(\xi)}{d\xi}\right]\frac{\partial \xi}{\partial x} = \frac{d^2 f(\xi)}{d\xi^2} \tag{12.4}$$

$$\frac{\partial \varphi}{\partial t} = \frac{df(\xi)}{d\xi}\frac{\partial \xi}{\partial t} = a\frac{df(\xi)}{d\xi} \tag{12.5}$$

$$\frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial t}\right) = \frac{\partial}{\partial t}\left[a\frac{df(\xi)}{d\xi}\right] = \frac{d}{d\xi}\left[a\frac{df(\xi)}{d\xi}\right]\frac{\partial \xi}{\partial t} = a^2\frac{d^2 f(\xi)}{d\xi^2} \tag{12.6}$$

Substituting Eqs.(12.4) and (12.6) into Eq.(12.1) gives

$$\left(c^2 - a^2\right)\frac{d^2 f(\xi)}{d\xi^2} = 0 \tag{12.7}$$

from which we see that Eq.(12.2) will satisfy Eq.(12.1) if $a = \pm c$. This gives us two independent solutions, and, since Eq.(12.1) is linear, the sum of these two solutions will also be a solution. Thus,

$$\varphi(x,t) = f(x - ct) + g(x + ct) \tag{12.8}$$

is a solution of Eq.(12.1) for any choice of the functions f and g.

It is not difficult to show that all solutions of Eq.(12.1) can be written in the form of Eq.(12.8). This is shown by changing from the variables x and t in Eq.(12.1) to the new independent variables ξ and η with the following transformation equations:

$$\xi = x - ct \qquad (12.9)$$

$$\eta = x + ct \qquad (12.10)$$

Another application of the chain rule gives the following identities:

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\varphi}{\partial\eta}\frac{\partial\eta}{\partial x} = \frac{\partial\varphi}{\partial\xi} + \frac{\partial\varphi}{\partial\eta} \qquad (12.11)$$

$$\frac{\partial^2\varphi}{\partial x^2} = \frac{\partial}{\partial\xi}\left(\frac{\partial\varphi}{\partial\xi} + \frac{\partial\varphi}{\partial\eta}\right)\frac{\partial\xi}{\partial x} + \frac{\partial}{\partial\eta}\left(\frac{\partial\varphi}{\partial\xi} + \frac{\partial\varphi}{\partial\eta}\right)\frac{\partial\eta}{\partial x} = \frac{\partial^2\varphi}{\partial\xi^2} + 2\frac{\partial^2\varphi}{\partial\xi\partial\eta} + \frac{\partial^2\varphi}{\partial\eta^2} \qquad (12.12)$$

$$\frac{\partial\varphi}{\partial t} = \frac{\partial\varphi}{\partial\xi}\frac{\partial\xi}{\partial t} + \frac{\partial\varphi}{\partial\eta}\frac{\partial\eta}{\partial t} = -c\frac{\partial\varphi}{\partial\xi} + c\frac{\partial\varphi}{\partial\eta} \qquad (12.13)$$

$$\frac{\partial^2\varphi}{\partial t^2} = \frac{\partial}{\partial\xi}\left(-c\frac{\partial\varphi}{\partial\xi} + c\frac{\partial\varphi}{\partial\eta}\right)\frac{\partial\xi}{\partial t} + \frac{\partial}{\partial\eta}\left(-c\frac{\partial\varphi}{\partial\xi} + c\frac{\partial\varphi}{\partial\eta}\right)\frac{\partial\eta}{\partial t} = c^2\frac{\partial^2\varphi}{\partial\xi^2} - 2c^2\frac{\partial^2\varphi}{\partial\xi\partial\eta} + c^2\frac{\partial^2\varphi}{\partial\eta^2} \qquad (12.14)$$

Substituting Eqs.(12.12) and (12.14) into Eq.(12.1) gives the following result:

$$\frac{\partial^2\varphi}{\partial\xi\partial\eta} = 0 \qquad (12.15)$$

Integrating Eq.(12.15) twice gives the following solution for $\varphi(\xi,\eta)$:

$$\varphi(\xi,\eta) = f(\xi) + g(\eta) \qquad (\xi = x - ct,\ \eta = x + ct) \qquad (12.16)$$

Eq.(12.16) is identical with Eq.(12.8), and the expressions for ξ and η are called "characteristics" of Eq.(12.1). Although we will do nothing further with characteristics, there is a very general numerical technique that uses characteristics for calculating numerical solutions in wave mechanics problems that are governed by hyperbolic partial differential equations. For example, the "method of characteristics" is sometimes used to obtain accurate numerical solutions of Eqs.(11.9) and (11.12) when calculating the movement of flood waves down rivers, and the movement of unsteady pressure waves through pipes, known as waterhammer, can also be calculated by using the method of characteristics.

There is a very well known solution of Eq.(12.1) that holds when its solution domain extends from $x = -\infty$ to $x = \infty$. The presence of the second-order time derivative in

Eq.(12.1) means that two initial conditions are required for a unique solution. Therefore, we assume that these two initial conditions are given by

$$\varphi(x,0)=F(x) \tag{12.17}$$

$$\frac{\partial\varphi(x,0)}{\partial t}=G(x) \tag{12.18}$$

where F(x) and G(x) are known specified functions. Substituting the general solution of Eq.(12.1), given by Eq.(12.8), into the left sides of Eqs.(12.17) and (12.18) leads to the following two equations for the unknown functions f and g:

$$f(x)+g(x)=F(x) \tag{12.19}$$

$$-f'(x)+g'(x)=\frac{G(x)}{c} \tag{12.20}$$

Differentiating Eq.(12.19) with respect to x gives

$$f'(x)+g'(x)=F'(x) \tag{12.21}$$

Adding and then subtracting Eqs.(12.20) and (12.21) leads to solutions for $f'(x)$ and $g'(x)$.

$$f'(x)=\frac{1}{2}F'(x)-\frac{1}{2c}G(x) \tag{12.22}$$

$$g'(x)=\frac{1}{2}F'(x)+\frac{1}{2c}G(x) \tag{12.23}$$

Thus, integrating Eqs.(12.22) and (12.23) gives expressions for f(x) and g(x).

$$f(x)=\frac{1}{2}F(x)-\frac{1}{2c}\int_0^x G(\xi)\,d\xi+A \tag{12.24}$$

$$g(x)=\frac{1}{2}F(x)+\frac{1}{2c}\int_0^x G(\xi)\,d\xi+B \tag{12.25}$$

where A and B are constants. Therefore, replacing x with x-ct in Eq.(12.24) and x with x+ct in Eq.(12.25) and adding the two equations gives the right side of Eq.(12.8).

$$\varphi(x,t)=\frac{1}{2}\left[F(x-ct)+F(x+ct)\right]+\frac{1}{2c}\left[\int_0^{x+ct}G(\xi)\,d\xi-\int_0^{x-ct}G(\xi)\,d\xi\right]+A+B \tag{12.26}$$

The initial condition (12.17) shows that $A+B=0$, and reversing the order of limits in the second integral by reversing the sign in front gives the final result.

$$\varphi(x,t)=\frac{1}{2}\left[F(x-ct)+F(x+ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct}G(\xi)\,d\xi \tag{12.27}$$

Eq.(12.27) is known as D'Alembert's solution for the wave equation and is found in many introductory texts dealing with partial differential equations.

Eq.(12.27) is a very elegant result, but the rather mechanical use of mathematics in its derivation sheds little light upon the physical behaviour of solutions of the wave equation. It is more instructive for this purpose to consider the general solution given by Eq.(12.8) when the g(x+ct) = 0. In this case, the solution

$$\varphi(x,t) = f(x-ct) \tag{12.28}$$

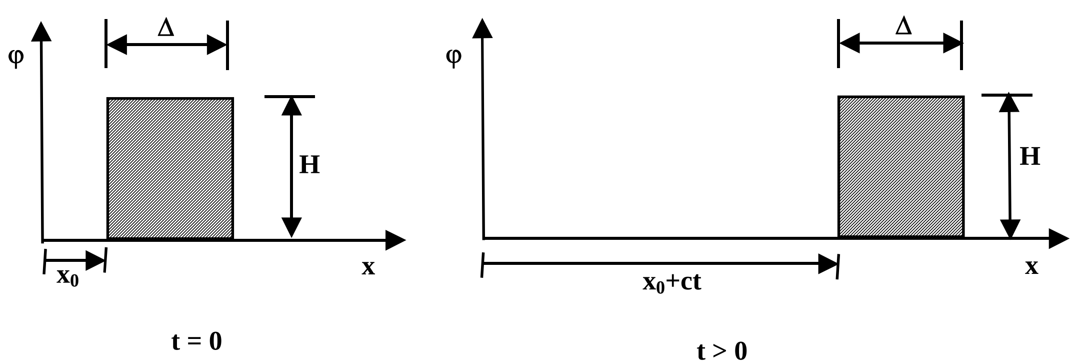
indicates that $\varphi(x,t)$ is constant when x-ct is constant. Thus, a point of constant φ on a wave has a location given by the equation

$$x(t) - ct = K \tag{12.29}$$

where K is a constant. Differentiation of Eq.(12.29) shows that the velocity of this point of constant φ is given by c.

$$\frac{dx(t)}{dt} = c \tag{12.30}$$

This shows that c is the speed or celerity of a wave, and it also shows that f(x-ct) gives the equation of a wave with constant shape translating in the positive x direction. This result is shown graphically with the following sketch of a translating square wave:



In the same way, it can be shown that

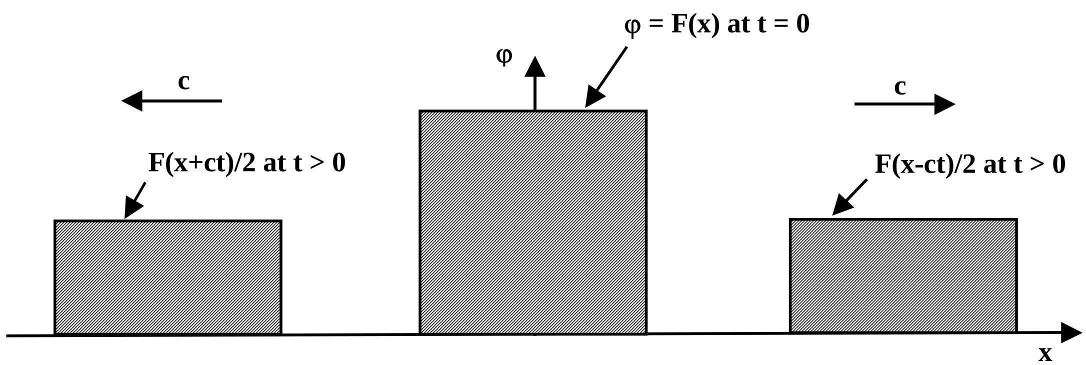
$$\varphi(x,t) = g(x+ct) \qquad (12.31)$$

is the equation of a wave translating with constant shape in the negative x direction. **Therefore, the general solution given by Eq.(12.8) is the sum or superposition of waves with constant shape travelling at speed c in both positive and negative x directions.**

Points discussed in the preceding paragraph make it relatively easy to interpret the result given by the first two terms on the right side of Eq.(12.27). If a wave starts from rest, in which case G(x) = 0 in Eqs.(12.18) and (12.27), then Eq.(12.27) reduces to

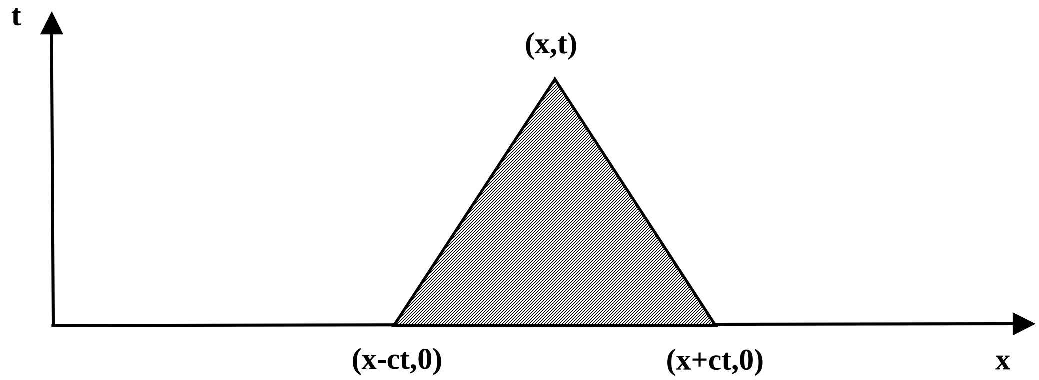
$$\varphi(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] \qquad (12.32)$$

If we consider F(x) to be a square wave, for example, then the following sketch illustrates the behaviour of the solution given by Eq.(12.32):



When t = 0, Eq.(12.32) reduces to $\varphi(x,0) = F(x)$, which is the square wave shown near the origin. When t > 0, Eq.(12.32) gives $\varphi(x,t)$ as the sum of the rightward travelling wave shown on the right in the above sketch and the leftward travelling wave shown on the left. These two travelling waves each have the same base width but half the height of the original square wave and travel with a speed c. Thus, the initial square wave at t = 0 is the superposition of the two travelling waves.

Another important consequence of Eq.(12.27) is seen by considering the following sketch in the (x,t) plane:

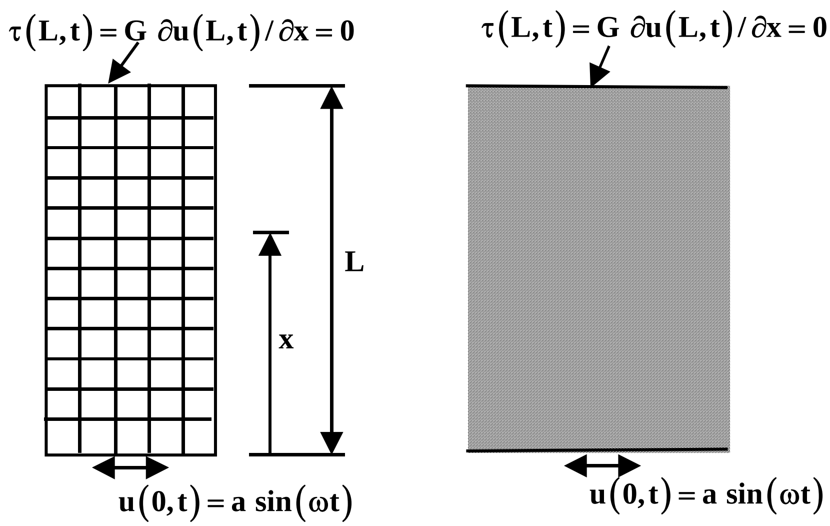


The apex of the triangle at (x,t) is the point where $\varphi(x,t)$ is calculated from Eq.(12.27), the sloping sides coincide with the characteristics of Eq.(12.1) and the triangle base coincides with the x axis between $x-ct$ and $x+ct$. It is seen from the above sketch and Eq.(12.27) that the value of φ at (x,t) depends only upon data along the base of the cross hatched triangle, which means that data outside this interval has no influence upon the value of $\varphi(x,t)$. [When the wave equation contains additional terms, as in Eq.(11.26), the method of characteristics can be used to show that the value of φ at (x,t) also depends upon data contained within the triangle.] This triangle is called the **domain of influence**, and the end result is sometimes described by saying that solutions of the wave equation have a "finite signal velocity." This is in sharp contrast to the behaviour of a solution of the heat conduction equation, which depends upon information given along the entire x axis from $x=-\infty$ to $x=\infty$. Since this would require $c=\infty$ in the above sketch, this result is sometimes alluded to by saying that the heat conduction or diffusion equation has an infinite signal velocity. This is despite the fact that Eq.(12.27) is not a solution of the heat conduction equation.

Lecture 13

Solution of an Important Wave Mechanics Problem

In this lecture we will consider a problem that has applications in two different areas of seismic engineering. The first application, sketched below at the left, uses a shear-beam model for a tall building subjected to horizontal oscillations at its base.



The second application, pictured at the right, models the upward movement of shear waves through a soft layer of earth. The following mathematical problem describes horizontal displacements, $u(x,t)$, for either application:

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad (0 < x < L, \; -\infty < t < \infty) \tag{13.1}$$

$$u(0,t) = a \sin(\omega t) \qquad (-\infty < t < \infty) \tag{13.2}$$

$$\frac{\partial u(L,t)}{\partial x} = 0 \qquad (-\infty < t < \infty) \tag{13.3}$$

Eq.(13.1) is the governing partial differential equation, Eq.(13.2) specifies a sinusoidal oscillation at the base and Eq.(13.3) requires a zero shear stress either at the top of the building or on the ground surface. If we wanted to calculate the complete time history of motion, starting from a state of rest, then we would also have to include two initial conditions. In this case, however, we are going to calculate what is often referred to as the "steady-state" solution, which is the solution that occurs after sufficient time has elapsed since the start of motion for the effect of initial conditions to disappear. This is often, but not always, done when solving wave mechanics problems, and it is the reason that the solution domain has the limits $-\infty < t < \infty$ rather than $0 < t < \infty$. (If motion starts at $t = -\infty$, then initial conditions have no effect upon u(x,t) at finite values of t.)

A suitable set of scaled dimensionless variables for this problem is given by

$$\left(u^*, x^*, t^*, \omega^*\right) = \left(\frac{u(x,t)}{a}, \frac{x}{L}, \frac{tc}{L}, \frac{\omega L}{c}\right) \qquad (13.4)$$

where the time scale, L/c, is the time required for a wave to travel from x = 0 to x = L. Introducing these dimensionless variables into Eqs.(13.1)-(13.3) gives the following set of equations:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad \left(0 < x < 1,\ -\infty < t < \infty\right) \qquad (13.5)$$

$$u(0,t) = \sin(\omega t) \qquad \left(-\infty < t < \infty\right) \qquad (13.6)$$

$$\frac{\partial u(1,t)}{\partial x} = 0 \qquad \left(-\infty < t < \infty\right) \qquad (13.7)$$

where the asterisk superscript has been omitted for notational convenience.

A big advantage of considering steady-state problems possessing boundary conditions with sinusoidal variations with time is that variables are easily separated into the product of a function of x with the sinusoidal function of time. In this case, Eq.(13.6) suggests that we look for a solution in the form

$$u(x,t) = \varphi(x)\sin(\omega t) \qquad (13.8)$$

Substituting Eq.(13.8) into Eqs.(13.5)-(13.7) gives the following set of equations to determine $\varphi(x)$:

$$\frac{d^2\varphi}{dx^2}+\omega^2\varphi=0 \qquad (0<x<1) \tag{13.9}$$

$$\varphi(0)=1 \tag{13.10}$$

$$\frac{d\varphi(1)}{dx}=0 \tag{13.11}$$

The general solution of Eq.(13.9) is given by

$$\varphi(x)=A\ \cos(\omega x)+B\ \sin(\omega x) \tag{13.12}$$

Boundary condition (13.10) requires A = 1, and boundary condition (13.11) gives $B=\sin(\omega)/\cos(\omega)$. Therefore, the problem solution is given by

$$u(x,t)=\varphi(x)\ \sin(\omega t)=\left[\cos(\omega x)+\frac{\sin(\omega)\ \sin(\omega x)}{\cos(\omega)}\right]\sin(\omega t) \tag{13.13}$$

However, Eq.(13.13) can be rewritten more compactly as follows:

$$u(x,t)=\frac{\cos(\omega)\ \cos(\omega x)+\sin(\omega)\ \sin(\omega x)}{\cos(\omega)}\ \sin(\omega t)\equiv\frac{\cos\left[\omega(1-x)\right]}{\cos(\omega)}\ \sin(\omega t) \tag{13.14}$$

where we have used the identity cos(A-B) = cos(A) cos(B) + sin(A) sin(B).

It is of interest to point out that the trigonometric identity

$$\cos(A)\sin(B)=\frac{1}{2}\left[\sin(A+B)+\sin(A-B)\right]$$

can be used to rewrite Eq.(13.14) in the form

$$u(x,t)=\frac{\sin\left[\omega(t+x-1)\right]+\sin\left[\omega(t-x+1)\right]}{2\ \cos(\omega)} \tag{13.15}$$

which, in agreement with Eq.(12.8), shows that the solution is the sum of upward and downward travelling waves. However, the result that is of most engineering significance follows from using Eq.(13.14) to calculate the ratio $u(x,t)/u(0,t)$, where $u(0,t)=\sin(\omega t)$.

$$\frac{u(x,t)}{u(0,t)}=\frac{\cos\left[\omega(1-x)\right]}{\cos(\omega)} \tag{13.16}$$

The left side of Eq.(13.16) is the ratio of the displacement at any point in the building or soil to the forcing displacement at x = 0. The absolute magnitude of this ratio is called the wave amplification, and Eq.(13.16) shows that wave amplification becomes infinite when $\left|\cos(\omega)\right|$ becomes zero. Since ω actually stands for $\omega^* = \omega L / c$, we see that resonance occurs when

$$\frac{\omega L}{c} = (2n-1)\frac{\pi}{2} \qquad (n = 1,\ 2,\ 3,\ \cdots) \tag{13.17}$$

Thus, there are an infinite number of resonance frequencies. Contrast this with the behaviour of a mass suspended from a spring, which has only one resonance frequency.

We will finish this lecture by a discussion of why the solution obtained herein is often referred to as a "steady-state" solution, even though it still depends upon time. The discussion assumes that any physical system has at least some damping and, therefore, is described by Eq.(11.26). An example is given by the following problem:

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + 2\varepsilon \frac{\partial u}{\partial t} + f \qquad (0 < x < L,\ 0 < t < \infty) \tag{13.18}$$

$$u(0,t) = g \qquad (0 < t < \infty) \tag{13.19}$$

$$\frac{\partial u(L,t)}{\partial x} = h \qquad (0 < t < \infty) \tag{13.20}$$

$$u(x,0) = F(x) \qquad (0 < x < L) \tag{13.21}$$

$$\frac{\partial u(x,0)}{\partial t} = G(x) \qquad (0 < x < L) \tag{13.22}$$

where f, g and h may be either constants or sinusoidal functions of time and ε is a positive damping coefficient.

A steady-state solution, u_{ss}, is defined to be a solution of Eqs.(13.18)-(13.20). (i.e. The initial conditions (13.21) and (13.22) are ignored when calculating u_{ss}.) If f, g and h are constants, then u_{ss} will not depend upon time. If one or more of the functions f, g and h vary linearly with $\sin(\omega t)$ and/or $\cos(\omega t)$, then u_{ss} can be calculated with the methods used in this lecture. In this case, u_{ss} will be proportional to some linear combination of $\sin(\omega t)$ and $\cos(\omega t)$, and the "steady-state" solution will depend upon time. In either case, since the steady-state solution satisfies both boundary conditions and the inhomogeneous form of Eq.(13.18), a solution for $u(x,t)$ can be found in the form

$$u(x,t) = u_{ss}(x,t) + \varphi(x,t) \tag{13.23}$$

where $\varphi(x,t)$ satisfies the homogeneous forms of Eqs.(13.18), (13.19) and (13.20). This means, for this particular problem, that $\varphi(x,t)$ can be represented by the following Fourier series:

$$\varphi(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\alpha_n \frac{x}{L}\right) \qquad \left[\alpha_n = (2n-1)\frac{\pi}{2}\right] \tag{13.24}$$

Substituting $\varphi(x,t)$ from Eq.(13.24) into the homogeneous form of Eq.(13.18) shows that $a_n(t)$ is a solution of the following ordinary differential equation:

$$a_n'' + 2\varepsilon a_n' + c^2\alpha_n^2 a_n = 0 \tag{13.25}$$

which has the general solution

$$a_n(t) = e^{-\varepsilon t}\left[A_n \cos\left(t\sqrt{(\alpha c)^2 - \varepsilon^2}\right) + B_n \sin\left(t\sqrt{(\alpha c)^2 - \varepsilon^2}\right)\right] \tag{13.26}$$

The constants A_n and B_n can be determined from the initial conditions given by Eqs.(13.21) and (13.22), but the important point to notice is that both $a_n(t)$ and $\varphi(x,t)$ vanish as $t \to \infty$ because ε is positive. Therefore, if we first take the limit $t \to \infty$ and then take the limit $\varepsilon \to 0$, we find that the steady-state solution satisfies the following set of equations:

$$c^2 \frac{\partial^2 u_{ss}}{\partial x^2} = \frac{\partial^2 u_{ss}}{\partial t^2} + f \qquad (0 < x < L,\ 0 < t < \infty) \tag{13.27}$$

$$u_{ss}(0,t) = g \qquad (0 < t < \infty) \tag{13.28}$$

$$\frac{\partial u_{ss}(L,t)}{\partial x} = h \qquad (0 < t < \infty) \tag{13.29}$$

This suggests that a steady-state solution is the solution that occurs as $t \to \infty$ in a physical system that has a small amount of damping. Damping causes the effect of initial conditions upon the solution to be “forgotten” at large values of time.

Lecture 14

The Solution of Laplace's Equation

Laplace's equation, which is the simplest form of an elliptic equation, describes many different kinds of problems that interest civil engineers. For example, the Laplace equation describes problems in steady groundwater flow, heat conduction, contaminant diffusion in a motionless fluid and stresses in an elastic bar subjected to pure torsion. However, one of the simplest and best known applications concerns irrotational flow for incompressible fluid motion. The governing equation follows from a control volume statement of the continuity equation

$$\iint_S \overline{V} \cdot \hat{n} \; dS = 0 \tag{14.1}$$

where $\overline{V} =$ fluid velocity vector, S = control volume surface (closed) and $\hat{n} =$ outward unit normal on S. An application of the divergence theorem gives

$$\iiint_{Vol} \overline{\nabla} \cdot \overline{V} \; dx\; dy\; dz = 0 \tag{14.2}$$

where Vol = the control volume enclosed by the control surface, S. Since Eq.(14.2) holds for arbitrary choices of Vol, we obtain the following partial differential equation form of the continuity equation:

$$\overline{\nabla} \cdot \overline{V} = 0 \tag{14.3}$$

which holds at all points in an incompressible flow where Eq.(14.1) holds. If it is further assumed that the velocity field can be generated from the positive gradient of a potential function, φ, then

$$\overline{V} = \overline{\nabla}\varphi \tag{14.4}$$

A few authors, in analogy with problems in groundwater flow, heat conduction and diffusion, place a minus sign in front of the right side of Eq.(14.4). However, most authors choose to use the positive gradient of φ, as shown in Eq.(14.4). Elimination of $\overline{V}$ from Eqs.(14.3) and (14.4) shows that φ is a solution of Laplace's equation.

$$\nabla^2\varphi \equiv \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \qquad (14.5)$$

Once a solution for φ has been found from Eq.(14.5), $\overline{V}$ can be found from Eq.(14.4) and inserted into the Bernoulli equation to calculate pressures:

$$h + \frac{\overline{V} \cdot \overline{V}}{2g} + \frac{1}{g}\frac{\partial\varphi}{\partial t} = H \qquad (14.6)$$

where h = piezometric head, g = acceleration of gravity and H = constant. Many texts, such as Hunt (1995, chapter 6), give a derivation of Eq.(14.6).

We will be concerned with solving Eq.(14.5) for two-dimensional flows in the (x,y) plane, in which case Eq.(14.5) reduces to

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \qquad (14.7)$$

If, as in lecture 12, we search for a solution of Eq.(14.7) in the form

$$\varphi(x, y) = f(\xi) \qquad (\xi = x + ay) \qquad (14.8)$$

then use of the chain rule of differential calculus gives the result

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = \left(1 + a^2\right)\frac{d^2 f(\xi)}{d\xi^2} = 0 \qquad (14.9)$$

from which we conclude that $a = \pm\sqrt{-1} = \pm i$ and

$$\varphi(x, y) = f(z) + g(\overline{z}) \qquad \left(z = x + iy,\ \overline{z} = x - iy\right) \qquad (14.10)$$

where z and $\overline{z}$ are the almost universal notation for a complex variable and its conjugate, respectively [rather than a third spatial coordinate, as in Eq.(14.5)]. If the independent variables x and y in Eq.(14.7) are transformed to z and $\overline{z}$ by use of the transformation equations given in (14.10), then Eq.(14.7) takes the following form:

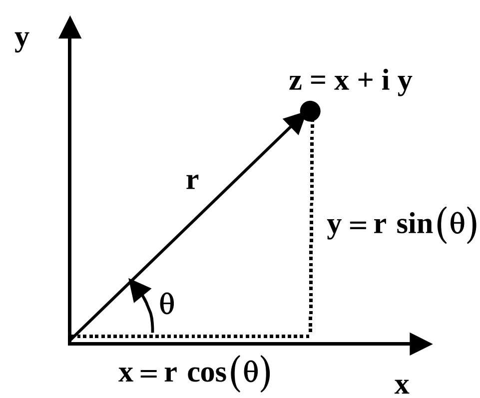
$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 4\frac{\partial^2\varphi}{\partial z\partial\overline{z}} = 0 \qquad (14.11)$$

Since Eq.(14.10) gives the general solution of Eq.(14.11), we conclude that all solutions of Eq.(14.7) can be written in the form given by Eq.(14.10). In other words, Eq.(14.10) gives the general solution of Eq.(14.7). Furthermore, since $\varphi(x, y)$ is a real valued function, we also conclude that $g(\bar{z})$ is the complex conjugate of f(z). This means that the general solution of Eq.(14.7) when $\varphi(x,y)$ is a real function can always be written in the form

$$\varphi(x,y) = \frac{1}{2}\left[f(z) + \overline{f(z)}\right] \qquad (z = x + iy) \tag{14.12}$$

where f(z) in Eq.(14.10) has been redefined as f(z)/2. In other words, the general solution of Laplace's equation is the real part of a complex function f(z). Therefore, it is only necessary to study the behaviour of f(z) in Eq.(14.12), and not the behaviour of $g(\bar{z})$ in Eq.(14.10), when using complex variables to solve for φ.

A complex variable, z, has a real part, x, and an imaginary part, y, and is written in the form z = x + iy. In polar coordinates, where $x = r\cos(\theta)$ and $y = r\sin(\theta)$, this can also be written as $z = r\cos(\theta) + i\, r\sin(\theta) = r\left[\cos(\theta) + i\sin(\theta)\right]$. A graphical representation of point z in the complex z plane is shown in the following sketch:



Euler's formula gives one of the most useful ways to represent a complex number:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{14.13}$$

The identity given by Eq.(14.13) can be derived by setting $z = i\theta$ in the Taylor series expansion $e^z = 1 + z/1! + z^2/2! + z^3/3! + \cdots$ to obtain

$$e^{i\theta} = 1 + \frac{(i\theta)}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \cdots \qquad (14.14)$$

Since $i^2 = -1,\ i^3 = ii^2 = -i,\ i^4 = i^2 i^2 = 1,$ etc. we can group real and imaginary parts to obtain

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \qquad (14.15)$$

The real part of the right side of Eq.(14.15) is the Taylor series expansion of $\cos(\theta)$ for $-\infty < \theta < \infty$, and the imaginary part is the expansion of $\sin(\theta)$ for $-\infty < \theta < \infty$. Thus, Eq.(14.15) is identical with the formula given in Eq.(14.13). Furthermore, a more advanced concept known as the permanence of forms of functional identities [Churchill, 1960, 261-262] can be used to show that Eq.(14.13) holds even when θ is a complex number with real and imaginary parts.

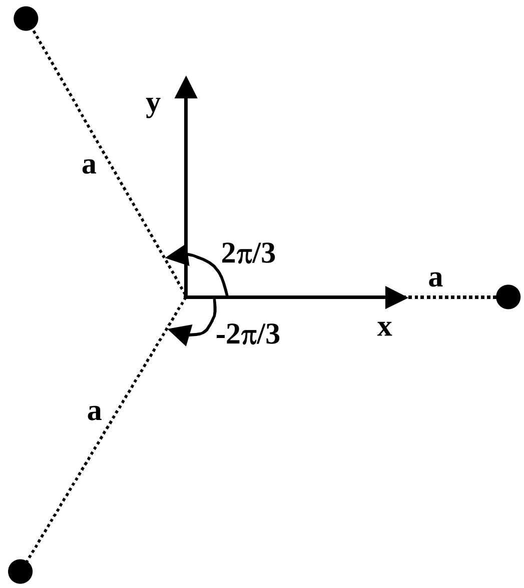
An interesting application of Eq.(14.13) can be illustrated by solving the equation $z^3 - a^3 = 0$. Since $e^{\pm i2n\pi} = \cos(2n\pi) \pm i\ \sin(2n\pi) = 1$ when n is an integer, this equation can be written as follows:

$$z^3 = a^3 = a^3 e^{i2n\pi} \qquad (n = \pm 0, 1, 2, 3, 4 \cdots) \qquad (14.16)$$

Taking the one-third power of both sides of Eq.(14.16) gives

$$z = ae^{i2n\pi/3} \qquad (n = \pm 0, 1, 2, 3, 4 \cdots) \qquad (14.17)$$

At first glance it appears that Eq.(14.17) gives an infinite number of roots for z. However, more careful consideration shows that only three of these roots plot in different locations in the z plane, as shown below:



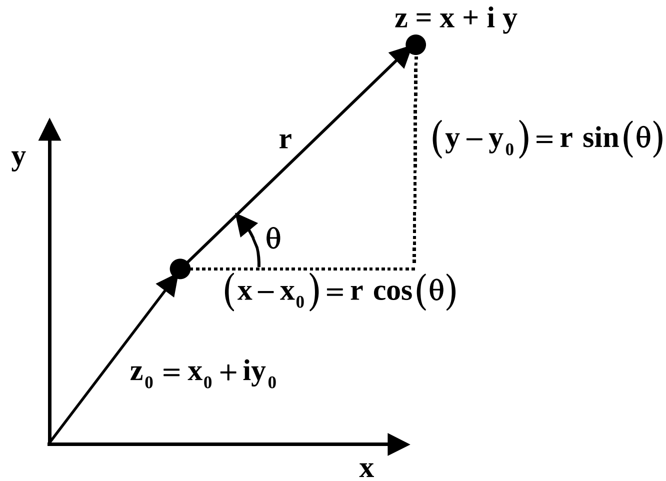
The complex numbers corresponding to these three locations are

$$z = a \qquad (n = 0) \tag{14.18}$$

$$z = a\ e^{i2\pi/3} = a\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right] = a\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \qquad (n = 1) \tag{14.19}$$

$$z = a\ e^{-i2\pi/3} = a\left[\cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)\right] = a\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \qquad (n = -1) \tag{14.20}$$

Inserting any other positive or negative integer for n in Eq.(14.17) gives a complex number with the same real and imaginary parts as one of the three numbers given above.



Another application of Eq.(14.13) that occurs frequently in applications concerns the use of polar coordinates to represent the difference of two complex numbers.

$$z - z_0 = (x - x_0) + i(y - y_0) \qquad (z = x + iy,\ z_0 = x_0 + iy_0) \qquad (14.21)$$

The above sketch shows that this difference can be rewritten as follows:

$$z - z_0 = r\ e^{i\theta} \qquad \left[r = \sqrt{(x - x_0)^2 + (y - y_0)^2},\ \theta = \tan^{-1}\left(\frac{y - y_0}{x - x_0}\right) \right] \qquad (14.22)$$

Lecture 15

Analytic Functions of a Complex Variable

Substituting z = x + iy in any function w = f(z) allows it to be separated into real and imaginary parts that are often denoted by $\varphi(x,y)$ and $\psi(x,y)$, respectively.

$$w = f(z) = \varphi(x,y) + i\psi(x,y) \qquad (z = x + iy) \tag{15.1}$$

In analogy with the calculus of real variables, the derivative of f(z) is defined as

$$\frac{df(z)}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \tag{15.2}$$

This means that any of the formulae used to calculate derivatives of real functions carry over to complex variables. For example, the derivative of z^k is $k z^{k-1}$, and the derivative of $\cos(kz)$ is $-k\sin(kz)$.

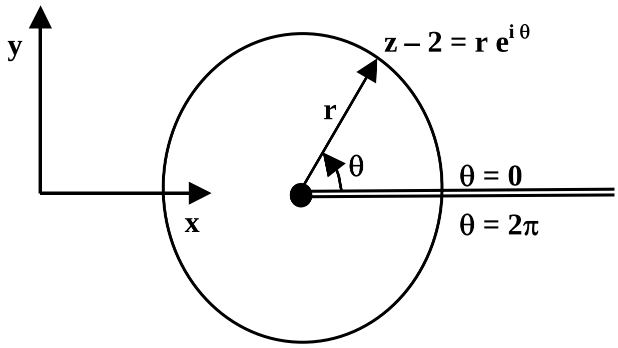
Although a function f(z) may sometimes be described as analytic, it is more accurate to say that a function, f(z), is analytic within some particular region. This is because a constant is the only "function" that is analytic at all points in the z plane. (Note: "All points in the z plane" includes the point $z = \infty$, which may seem puzzling at first. However, this should become clear later.) In general, a function is analytic within a region if it is single-valued (continuous) and finite at every point in the region and if its derivative is single-valued and finite at every point in the region. These ideas are probably explained best by considering particular examples for f(z).

For a very simple first example, we will examine the behaviour of the function $f(z) = z^3 = \left(r e^{i\theta}\right)^3 = r^3 e^{i3\theta} = r^3\left[\cos(3\theta) + i\sin(3\theta)\right]$. If we consider the continuity of f(z) within a circle surrounding the origin, and if we choose θ to be within the range

$-\pi \le \theta \le \pi$[12], then the only place that f(z) might become discontinuous is along the ray that has $\theta = -\pi$ and $\theta = \pi$. (This is the same ray in the z plane, although the specified range for θ means that θ becomes discontinuous along this ray.) Along $\theta = -\pi$ we calculate $f(z) = r^3\left[\cos(-3\pi) + i\sin(-3\pi)\right] = -r^3$, and along $\theta = \pi$ we calculate $f(z) = r^3\left[\cos(3\pi) + i\sin(3\pi)\right] = -r^3$. Thus, f(z) has the same values on this ray, regardless of whether the ray is approached through positive values of θ or through negative values of θ. This means that $f(z) = z^3$ is single-valued at all points within a circle surrounding the origin. Similar considerations show that $f'(z) = 3z^2$ is also single-valued and finite at all points within this circle. On the other hand, both $f(z)$ and $f'(z)$ become infinite as z becomes infinite. Therefore, we conclude that $f(z) = z^3$ is analytic everywhere in the finite z plane but ceases to be analytic at infinity. (The "point" at infinity consists of all points on the circumference of a circle as the radius of the circle becomes infinite A closer examination of any function, f(z), at infinity is carried out by replacing z with 1/t and then examining the function f(1/t) in the neighbourhood of t = 0.)

For a second example, consider $f(z) = \sqrt{z-2} \equiv r^{1/2}e^{i\theta/2}$ where $r = \sqrt{(x-2)^2 + y^2}$ and $\theta = \tan^{-1}\left[y/(x-2)\right]$. We will examine this function within a small circle surrounding the point z = 2 + i0 for $0 \le \theta \le 2\pi$, as shown in the sketch below:

[12] The range for θ can be chosen in any one of an infinite number of ways so long as θ has a maximum change of 2π as z traverses a circle about the origin. For example, we could have chosen $0 \le \theta \le 2\pi$ or $-3\pi/2 \le \theta \le \pi/2$.



The rays $\theta = 0$ and $\theta = 2\pi$ coincide in the z plane, but θ is discontinuous along this line. Therefore, we examine the behaviour of f(z) along this line. When $\theta = 0$, we calculate $f(z) = r^{1/2}e^{i0} = r^{1/2}$. On the other hand, if we set $\theta = 2\pi$, then we calculate $f(z) = r^{1/2}e^{i\pi} = r^{1/2}\left[\cos(\pi) + i\sin(\pi)\right] = -r^{1/2}$. Therefore, f(z) is discontinuous along the positive real axis for $x > 2$, and f(z) is not analytic at points along this line, which is called a "branch cut."

A branch point is a point within a region where f(z) becomes discontinuous when z moves continuously around a small closed circle that surrounds the branch point. In the previous example, the point z = 2 was a branch point since f(z) failed to return to its starting value of $r^{1/2}$ as point z moved from $\theta = 0$ to $\theta = 2\pi$ along the circumference of a small circle surrounding z = 2. The point at $z = \infty$ can be examined by setting z = 1 / t to obtain $f(1/t) = \sqrt{1/t - 2} \sim 1/\sqrt{t}$ as $t \to 0$. (The symbol $\sim$ means "asymptotically equal to.") Therefore, since $1/\sqrt{t}$ also fails to return to its starting value when point t completes a closed circuit about the point t = 0, we see that a second branch point exists at $z = \infty$ (which corresponds to the point t = 0). Thus, the branch cut for our example, which was the real axis for $2 < x < \infty$, joined the branch point at z = 2 to the branch point at $z = \infty$. **In fact, a branch cut, which is a line or cut inserted in the z plane to keep f(z) single-valued within a region, always joins two branch points.**

The ideas just discussed allow us to make a few important observations about the function

$$f(z) = (z - z_0)^k \tag{15.3}$$

First, if k is a positive integer, then f(z) is analytic everywhere except at the point $z = \infty$. Second, if k = 0, then f(z) is analytic everywhere. Third, if k is a negative integer, then f(z) is analytic everywhere except at the point $z = z_0$. Fourth, if k is a positive or negative number that is not an integer, then f(z) is analytic only within a region that excludes the two branch points at $z = z_0$ and $z = \infty$ and a branch cut that joins these two branch points.

A second type of fundamental singularity is given by a logarithm. For example, the function

$$f(z) = \ln(z-2) = \ln\left(r\,e^{i\theta}\right) = \ln(r) + \ln\left(e^{i\theta}\right) = \ln(r) + i\theta \quad \left[r = \sqrt{(x-2)^2 + y^2},\ \theta = \tan^{-1}\left(\frac{y}{x-2}\right)\right] \tag{15.4}$$

has branch points at z = 2 and $z = \infty$ and a branch cut consisting of any curved or straight line joining these two branch points. Typically, we could choose the same branch cut shown in the previous sketch, in which case we would require $0 \le \theta \le 2\pi$, or we could choose the portion of the x axis extending from x = 2 to $x = -\infty$, in which case we would specify $-\pi \le \theta \le \pi$. Then the function f(z) would be analytic everywhere except at the two branch points and along the branch cut.

More complicated functions for f(z) are handled by using series expansions to examine behaviours near singularities. For example, consider the following function:

$$f\left(z\right)=\frac{\sqrt{z^2+1}}{\sin\left(z\right)} \qquad (15.5)$$

Since $\sqrt{z^2+1}=\sqrt{\left(z+i\right)\left(z-i\right)}$, we see that f(z) has branch points at z = i and z = -i. Setting z =1 / t , we find that $\sqrt{1/t^2+1}\sim 1/t$ near t = 0. Thus, the point at $z=\infty$ is a singularity that must be excluded from the region in which f(z) is analytic, but this point is not a branch point since f(1/t) is single-valued near t = 0.[13] Singularities also occur in f(z) where $\sin\left(z\right)=0$, at which points $z=n\pi+i0$ for $n=\pm 0,1,2,3\cdots$.[14] The leading term in a Taylor series expansion at $z=n\pi$ gives $\sin\left(z\right)\sim\left(-1\right)^n\left(z-n\pi\right)$, which means that these singular points must be excluded from the region in which f(z) is analytic. However, these singularities are not branch points and, therefore, are not joined with branch cuts. Typical ways in which f(z) can be defined so that it is analytic in regions away from its singularities, branch points and branch cuts are shown below.

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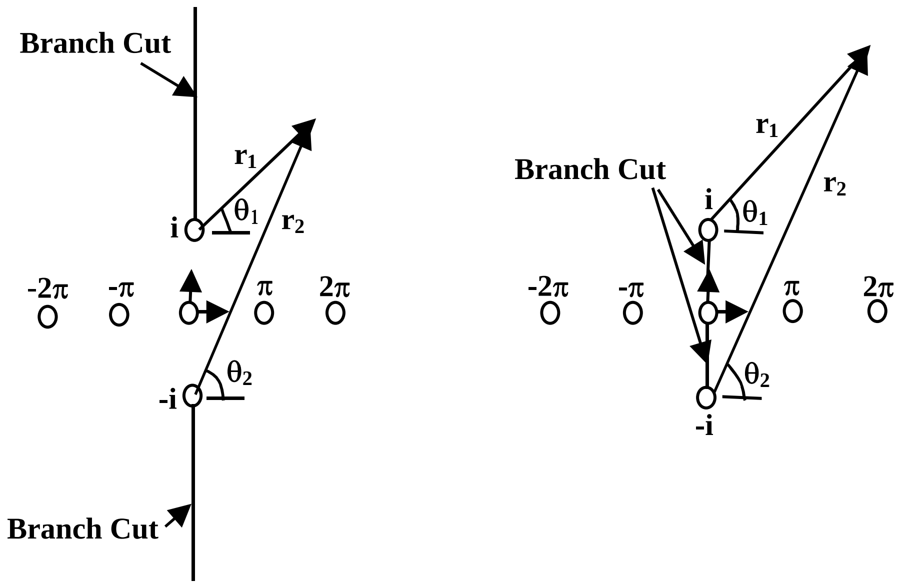
[13] Since $\sin(z)=\sum_{n=1}^{\infty}\left(-1\right)^{n+1}\frac{z^{2n-1}}{\left(2n-1\right)!} \quad \left(0\le\left|z\right|<\infty\right)$, we see that $\sin\left(1/t\right)$ has an expansion about t = 0 with an infinite number of negative integer subscripts. Using this result to expand f(1/t) about t = 0 shows that the expansion of f(1/t) about t = 0 also has an infinite number of negative integer subscripts. This is the definition of an isolated essential singularity at t = 0. Thus, f(z) has an isolated essential singularity at $z=\infty$, which means that it is singular but single-valued . Therefore, although $z=\infty$ must be excluded from the region in which f(z) is analytic, the point $z=\infty$ is not a branch point since f(z) is single-valued in the neighbourhood of $z=\infty$.

[14] Zill and Cullen (1992) show on pages 968-969 that these are the only zeros of sin(z).



Small circles show the location of singularities and branch points that must be excluded from the region in which f(z) is analytic. Branch cuts are shown with straight lines and must also be excluded from the region in which f(z) is analytic. Branch cuts shown in the left sketch are satisfactory choices if θ_1 for $z - i = r_1 e^{i\theta_1}$ lies in the range $-3\pi/2 \le \theta_1 \le \pi/2$ and θ_2 for $z + i = r_2 e^{i\theta_2}$ lies in the range $-\pi/2 \le \theta_2 \le 3\pi/2$. The branch cut shown in the sketch on the right coincides with the y axis for $-1 < y < 1$ and will be satisfactory if θ_1 and θ_2 both lie in the range $-\pi/2 \le \theta \le 3\pi/2$. (However, we could also specify the range $-3\pi/2 \le \theta \le \pi/2$ for θ_1 and θ_2.)

The previous problem shows that the process of determining a region in which a function f(z) is analytic requires two steps. First, all singularities, branch points and branch cuts must be located. Second, ranges must be specified for arguments (values of θ) at each branch point that will keep f(z) single-valued. Choices for branch cut locations and argument ranges are never unique. However, branch cuts are often chosen to coincide with physical boundaries. Thus, if f(z) in the above problem gave the solution for flow through a slot extending from z = -i to z = i, we would choose the branch cuts shown in the left sketch. On the other hand, if f(z) gave the solution for flow past a flat plate extending from z = -i to z = i, we would choose the branch cut shown in the right hand sketch. All of this would be done to ensure that f(z) remained analytic within the field of flow.

Lecture 16

The Cauchy-Riemann Equations and the Stream Function

Eqs.(15.1) and (15.2) are reproduced below:

$$w = f(z) = \varphi(x,y) + i\psi(x,y) \qquad (z = x + iy) \tag{16.1}$$

$$\frac{df(z)}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \tag{16.2}$$

Since $\Delta z = \Delta x + i\,\Delta y$, Eq.(16.2) can be rewritten as follows:

$$\frac{dw}{dz} = \lim_{\Delta x, \Delta y \to 0} \frac{\left[\varphi(x+\Delta x, y+\Delta y) + i\,\psi(x+\Delta x, y+\Delta y)\right] - \left[\varphi(x,y) + i\,\psi(x,y)\right]}{\Delta x + i\,\Delta y} \tag{16.3}$$

Since x and y are independent variables, the point $(x+\Delta x, y+\Delta y)$ can approach the point (x, y) along any path that we care to choose. Therefore, if these two points approach each other along a line parallel to the x axis, so that y is constant and $\Delta y = 0$, Eq.(16.3) reduces to

$$\frac{dw}{dz} = \lim_{\Delta x \to 0}\left[\frac{\varphi(x+\Delta x, y) - \varphi(x,y)}{\Delta x} + i\,\frac{\psi(x+\Delta x, y) - \psi(x,y)}{\Delta x}\right] = \frac{\partial\varphi}{\partial x} + i\frac{\partial\psi}{\partial x} \tag{16.4}$$

Alternatively, if the two points approach each other along a line parallel to the y axis, so that x is constant and $\Delta x = 0$, Eq.(16.3) reduces to

$$\frac{dw}{dz} = \lim_{\Delta y \to 0}\left[\frac{\varphi(x, y+\Delta y) - \varphi(x,y)}{i\,\Delta y} + i\,\frac{\psi(x, y+\Delta y) - \psi(x,y)}{i\,\Delta y}\right] = \frac{1}{i}\left[\frac{\partial\varphi}{\partial y} + i\frac{\partial\psi}{\partial y}\right] \tag{16.5}$$

However, if w is analytic at the point (x,y), then these two expressions for dw/dz must give exactly the same result. Since two complex functions are equal if, and only if, their real and imaginary parts are equal, this gives the following two equations:

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y} \tag{16.6}$$

$$\frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x} \tag{16.7}$$

Eqs.(16.6) and (16.7) are known as the Cauchy-Riemann equations, and this development shows that they must be satisfied at any point where w is an analytic function of the complex variable z.

The significance of the Cauchy-Riemann equations starts to become apparent when they are used to obtain partial differential equations that φ and ψ must satisfy. For example, differentiating Eq.(16.6) with respect to x and then using Eq.(16.7) to eliminate ψ gives

$$\frac{\partial^2\varphi}{\partial x^2}=\frac{\partial^2\psi}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial x}\right)=\frac{\partial}{\partial y}\left(-\frac{\partial\varphi}{\partial y}\right)=-\frac{\partial^2\varphi}{\partial y^2} \tag{16.8}$$

Thus, φ is a solution of the Laplace equation.

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0 \tag{16.9}$$

A similar procedure can be used to eliminate φ and show that ψ also satisfies the Laplace equation.

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0 \tag{16.10}$$

This means that real and imaginary parts of a function f(z) are solutions of Laplace's equation in any region where f(z) is analytic. It also means that we are free to define the real part of f(z), $\varphi(x,y)$, to be the velocity potential function for an irrotational flow problem.

A second important result follows from remembering that $\overline{\nabla}\varphi$ and $\overline{\nabla}\psi$ are vectors normal to curves of constant φ and constant ψ, respectively. Computing the dot product of these two vectors and using the Cauchy-Riemann equations to eliminate ψ gives

$$\overline{\nabla}\varphi\cdot\overline{\nabla}\psi=\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y}=\frac{\partial\varphi}{\partial x}\left(-\frac{\partial\varphi}{\partial y}\right)+\frac{\partial\varphi}{\partial y}\left(\frac{\partial\varphi}{\partial x}\right)=0 \tag{16.11}$$

But the dot product of two vectors is zero only if the magnitude of one or both of the vectors is zero or if the vectors are orthogonal. Since Eqs.(16.4)-(16.7) can be used to show that

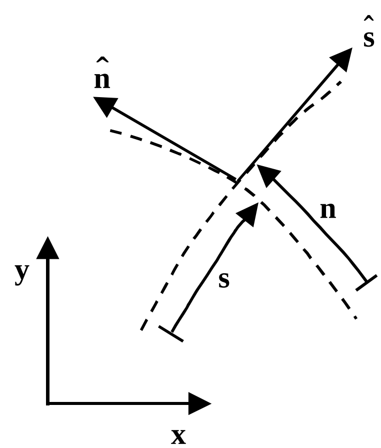
$$\left|\overline{\nabla}\varphi\right|^2=\left|\overline{\nabla}\psi\right|^2=\left|\frac{dw}{dz}\right|^2 \tag{16.12}$$

we see that normals to the curves of constant φ and constant ψ are orthogonal at every point where

$$0 < \left|\frac{dw}{dz}\right| < \infty \tag{16.13}$$

Since the angle between the normals to two curves also equals the angle between the curves, **we conclude that curves of constant φ and constant ψ are orthogonal at all points where Eq.(16.13) is satisfied.** Furthermore, since Eq.(14.4) shows the velocity vector in an irrotational flow is orthogonal to curves of constant φ at these points, and since streamlines are tangent to the velocity vector in a flow, **we also conclude that curves of constant ψ coincide with streamlines.** For this reason, the function $\psi(x, y)$ in Eq.(16.1) is called a stream function.

A third important consequence of the Cauchy-Riemann equations follows from consideration of the following sketch:



The two orthogonal curves in the sketch have unit tangents $\hat{s}$ and $\hat{n}$. Arc lengths s and n are measured from fixed points on these curves in the directions of $\hat{s}$ and $\hat{n}$, respectively, and direction cosines for the two unit tangents are given by

$$\hat{s} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} \tag{16.14}$$

$$\hat{n} = \frac{dx}{dn}\hat{i} + \frac{dy}{dn}\hat{j} \tag{16.15}$$

Furthermore, the cross product $\hat{k}\times\hat{s}=\hat{n}$ gives two relationships between the direction cosines.

$$\frac{dx}{ds}=\frac{dy}{dn} \tag{16.16}$$

$$\frac{dy}{ds}=-\frac{dx}{dn} \tag{16.17}$$

This allows the directional derivative of φ, in the direction of s, to be calculated in terms of the directional derivative of ψ, in the direction of n.

$$\frac{d\varphi}{ds}=\overline{\nabla}\varphi\cdot\hat{s}=\frac{\partial\varphi}{\partial x}\frac{dx}{ds}+\frac{\partial\varphi}{\partial y}\frac{dy}{ds}=\left(\frac{\partial\psi}{\partial y}\right)\left(\frac{dy}{dn}\right)+\left(-\frac{\partial\psi}{\partial x}\right)\left(-\frac{dx}{dn}\right)=\overline{\nabla}\psi\cdot\hat{n}=\frac{d\psi}{dn} \tag{16.18}$$

where use has been made of both the Cauchy-Riemann equations and Eqs.(16.16)-(16.17). If s is taken as arc length along a streamline, then $d\varphi/ds$ is the velocity magnitude on this streamline. Then the flow between any two streamlines is obtained from the following integral:

$$q=\int_{n_1}^{n_2}\overline{V}\cdot\hat{s}\,dn=\int_{n_1}^{n_2}\frac{d\varphi}{ds}\,dn=\int_{n_1}^{n_2}\frac{d\psi}{dn}\,dn=\psi_2-\psi_1 \tag{16.19}$$

The conclusion is that the flow rate through any region bounded by two streamlines is given by the difference in numerical values of ψ on the two streamlines.

A fourth consequence of the Cauchy-Riemann equations is obtained by using Eqs.(16.4) and (16.7).

$$\frac{dw}{dz}=\frac{\partial\varphi}{\partial x}+i\frac{\partial\psi}{\partial x}=\frac{\partial\varphi}{\partial x}-i\frac{\partial\varphi}{\partial y} \tag{16.20}$$

But $\overline{V}=\overline{\nabla}\varphi$ gives $u=\partial\varphi/\partial x$ and $v=\partial\varphi/\partial y$, where u and v are velocity components in the x and y directions, respectively. This allows Eq.(16.20) to be rewritten as follows:

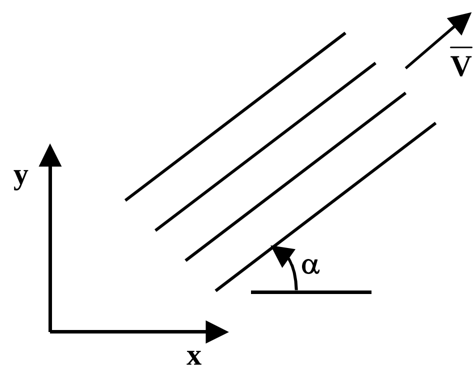
$$\frac{dw}{dz}=u-i\,v\equiv\sqrt{u^2+v^2}\left(\frac{u}{\sqrt{u^2+v^2}}-i\frac{v}{\sqrt{u^2+v^2}}\right)\equiv V\,e^{-i\theta} \tag{16.21}$$

where V = velocity vector magnitude and $\theta=$ angle that the velocity vector makes with the x axis. **Therefore, dw/dz gives the complex conjugate of the velocity vector for an irrotational flow.**

Lecture 17

Fundamental Solutions of Laplace's Equation

In this lecture we will use the theory developed in the previous lecture to obtain and discuss some fundamental solutions of Laplace's equation for irrotational flow. The first solution is for uniform flow inclined at an angle α to the x axis. A set of streamlines for this flow is sketched below:



Eq.(16.21) gives the complex velocity, dw/dz, for this flow as

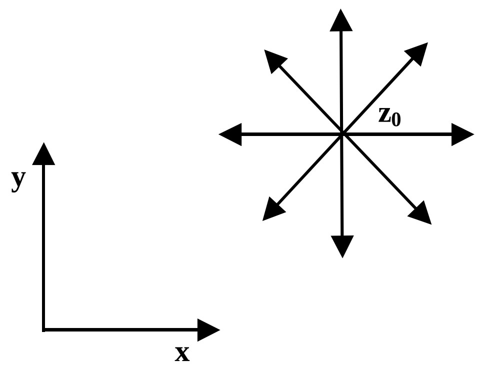
$$\frac{dw}{dz} = V\,e^{-i\alpha} \tag{17.1}$$

where V and α are constants since the flow is uniform. Integration of Eq.(17.1) gives the complex potential:

$$w = V\,e^{-i\alpha}\,z \tag{17.2}$$

where an additive constant has been omitted. [An integration constant can be added to any of the complex potentials considered herein. However, since velocities are calculated from the derivative of w(z), these constants are often omitted in irrotational flow problems. On the other hand, applications such as groundwater flow, heat conduction or contaminant diffusion sometimes require non-trivial additive constants. These are problems in which $\varphi(x, y)$ has some physical meaning, such as piezometric head, temperature or contaminant concentration, and an added constant is used to satisfy a boundary condition of specified φ.]

A solution in which flow is created and discharged radially outward from a point is called a source. A set of streamlines for a source located at the point z_0 is shown in the following sketch:



It is easy to verify that the solution for this flow is given by

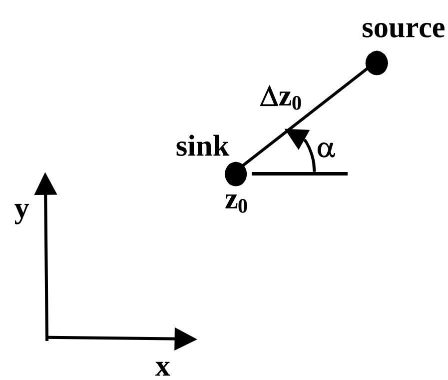
$$w = \frac{q}{2\pi}\ln\left(z - z_0\right) \qquad (17.3)$$

Since $w = \varphi + i\psi$, and since $\ln\left(z - z_0\right) = \ln\left(r\right) + i\theta$ where $r = \sqrt{\left(x - x_0\right)^2 + \left(y - y_0\right)^2}$ and $\theta = \tan^{-1}\left[\left(y - y_0\right)/\left(x - x_0\right)\right]$, we can equate imaginary parts on both sides of Eq.(17.3) to obtain $\psi = \left(\frac{q}{2\pi}\right)\theta$. Thus, radial lines of constant θ are lines of constant ψ and, therefore, coincide with streamlines. Furthermore, the difference in values of ψ on two adjacent streamlines after completing one circuit about the point z_0 is q (since θ changes by 2π after one complete circuit), which means that q is the flow rate emitted by the source. Finally, calculation of the complex velocity gives

$$\frac{dw}{dz} = u - iv = \frac{q}{2\pi}\frac{1}{\left(z - z_0\right)} = \frac{q}{2\pi r}e^{-i\theta} = \frac{q}{2\pi r}\left[\cos\left(\theta\right) - i\sin\left(\theta\right)\right] \qquad (17.4)$$

where we have set $(z - z_0) = r\,e^{i\theta}$. Eq.(17.4) shows that $u > 0$ and v = 0 when $\theta = 0$, which shows that the direction of flow is outward from z_0. A branch cut, consisting of a line extending in any direction from $z = z_0$ to $z = \infty$, must be used when calculating w from Eq.(17.3). On the other hand, the complex velocity given by Eq.(17.4) is single-valued but becomes infinite at $z = z_0$ since a finite flow passes through a circle with zero circumference at the source origin. The solution for a sink, where flow is absorbed at the point $z = z_0$, is obtained by replacing q with –q in Eqs.(17.3)-(17.4).

The potential for a doublet is obtained by placing a a source beside a sink and then letting the two singularities approach each other. This is shown in the following sketch:



The sink is placed at $z = z_0$, and the source is placed at $z = z_0 + \Delta z_0$ along the axis of the doublet, which is inclined at an angle α to the x axis. The mathematical equation for this flow is

$$w = \frac{q}{2\pi}\ln\left[z - \left(z_0 + \Delta z_0\right)\right] - \frac{q}{2\pi}\ln\left[z - z_0\right] \tag{17.5}$$

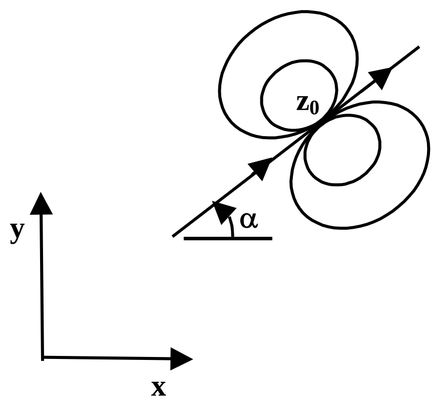
Multiply the right side of Eq.(17.5) by $\Delta z_0 / \Delta z_0$ and take the limit $\Delta z_0 \to 0$ to obtain

$$w = \lim_{\Delta z_0 \to 0}\left(\frac{q\,\Delta z_0}{2\pi}\right)\frac{\ln\left[z - \left(z_0 + \Delta z_0\right)\right] - \ln\left(z - z_0\right)}{\Delta z_0} \tag{17.6}$$

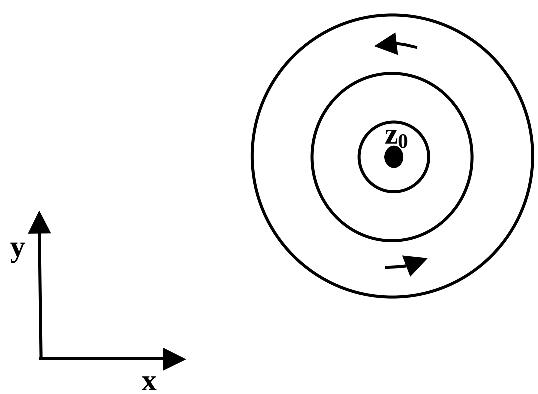
Since $\Delta z_0 = \left|\Delta z_0\right|\left[\cos(\alpha) + i\sin(\alpha)\right] = \left|\Delta z_0\right| e^{i\alpha}$, we can set $q\,\Delta z_0 = \left(q\left|\Delta z_0\right|\right)e^{i\alpha}$ to obtain the complex potential for a doublet.

$$w = \frac{\Delta e^{i\alpha}}{2\pi} \frac{\partial \ln\left(z - z_0\right)}{\partial z_0} = -\frac{\Delta}{2\pi} \frac{e^{i\alpha}}{\left(z - z_0\right)} \tag{17.7}$$

where $\Delta = q\left|\Delta z_0\right| =$ constant. The streamline pattern for a doublet is shown in the following sketch:



The streamline pattern for an irrotational vortex is a series of concentric circles, as shown below.



Since the streamlines and potential lines are identical with the potential lines and streamlines, respectively, for a source, we might expect the complex potential for a vortex to have the following form:

$$w = -i\frac{\Gamma_0}{2\pi} \ln\left(z - z_0\right) \tag{17.8}$$

Equating real and imaginary parts in Eq.(17.8) gives $\varphi = \frac{\Gamma_0}{2\pi}\theta$ and $\psi = -\frac{\Gamma_0}{2\pi}\ln(r)$, which shows that contours of constant ψ are concentric circles. Differentiating Eq.(17.8) gives the complex velocity.

$$\frac{dw}{dz} = u - i\,v = -i\frac{\Gamma_0}{2\pi}\frac{1}{(z - z_0)} = -i\frac{\Gamma_0}{2\pi}\frac{1}{r}e^{-i\theta} \tag{17.9}$$

Equating real and imaginary parts in Eq.(17.9) gives

$$u = -\frac{\Gamma_0}{2\pi}\frac{\sin(\theta)}{r} \tag{17.10}$$

$$v = \frac{\Gamma_0}{2\pi}\frac{\cos(\theta)}{r} \tag{17.11}$$

which shows that a positive value for Γ_0 corresponds to a vortex turning in the counter clockwise direction. Furthermore, since $V = \sqrt{u^2 + v^2} = \frac{\Gamma_0}{2\pi}\frac{1}{r}$, we see that the velocity magnitude decreases from infinity at r = 0 to zero at $r = \infty$. A quantity called the "circulation" is defined by the following integral:

$$\Gamma = \oint_C \overline{V} \cdot d\overline{r} \equiv \oint_C \overline{V} \cdot \frac{d\overline{r}}{ds}\, ds \equiv \oint_C \overline{V} \cdot \hat{t}\, ds \tag{17.12}$$

where $\hat{t} =$ the unit tangent to the closed curve C and ds = increment of arc length along C. If we choose C to coincide with any one of the circular streamlines of the vortex, then $\overline{V} \cdot \hat{t} = V = \Gamma_0 / (2\pi r)$ on C and Eq.(17.12) gives $\Gamma = \Gamma_0$. Therefore, Γ_0 in Eq.(17.8) is the circulation around a circle surrounding the vortex centre at $z = z_0$. It is not difficult to show that this same result holds when C is deformed into any other closed contour provided that the point z_0 remains within the contour.

Lecture 18

Superposition of Fundamental Solutions

The superposition principle states that the sum or difference of two or more solutions of a linear set of equations is also a solution of the same set of equations. This principle forms the basis for most analytical solutions in applied mathematics, and one reason that so few analytical solutions have been found for non-linear equations is that superposition cannot be used when any of the governing equations for a problem are non-linear. In this lecture we will superimpose some of the fundamental solutions obtained in the last lecture to find solutions for some new irrotational flow problems.

The first solution will be obtained by adding the potential for uniform flow in the positive x direction to the potential for flow from a source at the origin. Thus, setting $\alpha = 0$ in Eq.(17.2) and adding the result to the potential given by Eq.(17.3) gives

$$w = Vz + \frac{q}{2\pi}\ln(z) \tag{18.1}$$

Differentiating Eq.(18.1) gives the complex conjugate of the velocity vector.

$$\frac{dw}{dz} \equiv u - i\,v = V + \frac{q}{2\pi}\frac{1}{z} \tag{18.2}$$

Expressions for φ and ψ can be obtained by rewriting Eq.(18.1) as follows:

$$w \equiv \varphi + i\,\psi = V(x + i\,y) + \frac{q}{2\pi}\left[\ln(r) + i\,\theta\right] \tag{18.3}$$

Equating real and imaginary parts in Eq.(18.3) gives

$$\varphi = Vx + \frac{q}{2\pi}\ln(r) = Vx + \frac{q}{2\pi}\ln\left(\sqrt{x^2 + y^2}\right) \tag{18.4}$$

$$\psi = Vy + \frac{q}{2\pi}\theta = Vy + \frac{q}{2\pi}\tan^{-1}\left(\frac{y}{x}\right) \tag{18.5}$$

Eqs.(18.4)-(18.5) have been used to make a dimensionless plot of contours of constant φ and ψ in Fig. 18.1. This solution, which is usually referred to as flow past a half body, was made by noting from Eq.(18.2) that u = V and v = 0 as $z \to \infty$. Therefore, the product of the asymptotic width, B, of the half body at $x = \infty$ with V gives the flow, q, emitted from the source.

$$q = V B \tag{18.6}$$

Inserting this expression for q in Eqs.(18.4) and (18.5) leads to the following two equations:

$$\frac{\varphi}{VB} = \frac{x}{B} + \frac{1}{2\pi}\ln\left(\sqrt{x^2 + y^2}\right) \tag{18.7}$$

$$\frac{\psi}{VB} = \frac{y}{B} + \frac{1}{2\pi}\tan^{-1}\left(\frac{y}{x}\right) = \frac{y}{B} + \frac{1}{2\pi}\tan^{-1}\left(\frac{y/B}{x/B}\right) \tag{18.8}$$

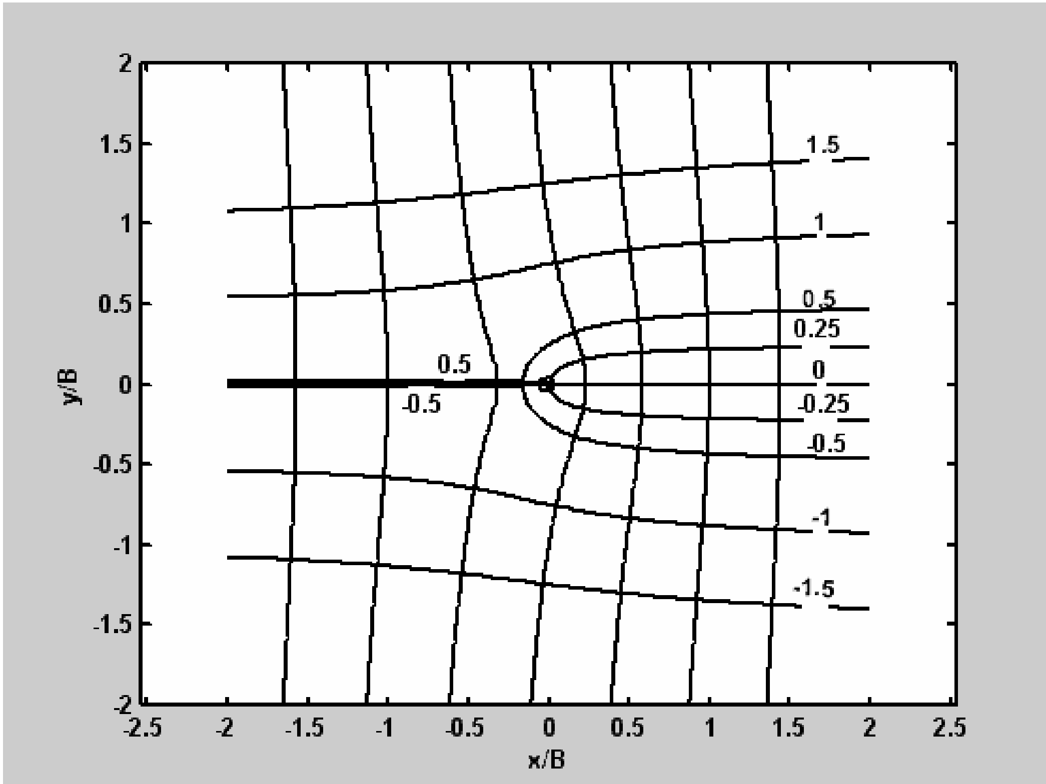
Subtracting the constant $\frac{1}{2\pi}\ln(B)$ from the right side of Eq.(18.7) puts the argument of the logarithm in dimensionless form so that the solution can be rewritten as follows:

$$\varphi = x + \frac{1}{2\pi}\ln\left(\sqrt{x^2 + y^2}\right) \tag{18.9}$$

$$\psi = y + \frac{1}{2\pi}\tan^{-1}\left(\frac{y}{x}\right) \tag{18.10}$$

where dimensionless variables are given as follows:

$$(\varphi^*, \psi^*, x^*, y^*) = \left(\frac{\varphi}{VB}, \frac{\psi}{VB}, \frac{x}{B}, \frac{y}{B}\right) \tag{18.11}$$



The asterisk superscript has been omitted in Eqs.(18.9)-(18.10) for notational convenience.

Fig. 18.1. Streamlines and potential lines for flow past a half body.

Numerical values for the stream function, ψ, have been shown immediately above the streamlines in Fig. 18.1. The streamlines labelled 0.5 and –0.5 in the right half of the figure form the half body boundary, which has a dimensionless width of unity at $x = \infty$. These same streamlines, $\psi = 0.5$ and $\psi = -0.5$, in the left half of the figure show a branch cut for the logarithm in Eq.(18.1). Note that streamlines and potential lines meet at right angles and that the difference in numerical values for ψ on upper and lower surfaces of the half body gives unity, which is the dimensionless flow emitted by the source (because no flow can cross the streamline that forms the half body surface).

Flow past a circular cylinder can be obtained by adding the complex potential for uniform flow in the positive x direction to the potential for a doublet at the origin

$$w = Vz + \frac{\Delta}{2\pi}\frac{1}{z} \tag{18.12}$$

where we have set $\alpha = \pi$ for the doublet axis so that $e^{i\alpha} = -1$ in Eq.(17.7). (The source has to be upstream from the sink. If instead you set $\alpha = 0$, then the signs will still work out because Δ will turn out to be a negative number.) The complex velocity is calculated from Eq.(18.12) as

$$\frac{dw}{dz} \equiv u - i\,v = V - \frac{\Delta}{2\pi}\frac{1}{z^2} \tag{18.13}$$

But u = v = 0 on the cylinder nose, where z = R + i0 and R is the cylinder radius. Inserting this requirement in Eq.(18.13) gives a value for the doublet strength, Δ.

$$\frac{\Delta}{2\pi} = V\,R^2 \tag{18.14}$$

Inserting Eq.(18.14) in Eqs.(18.12)-(18.13) gives the following equations, which have been written in dimensionless variables:

$$w = z + \frac{1}{z} \tag{18.15}$$

$$\frac{dw}{dz} \equiv u - i\,v = 1 - \frac{1}{z^2} \tag{18.16}$$

Dimensionless variables in Eqs.(18.15)-(18.16) are given by

$$\left(w^*, u^*, v^*, z^*\right) = \left(\frac{w}{VR}, \frac{u}{V}, \frac{v}{V}, \frac{z}{R}\right) \tag{18.17}$$

As usual, the asterisk superscript has been omitted in Eqs.(18.15)-(18.16) for notational convenience.

Setting $w = \varphi + i\,\psi$ in Eq.(18.15) and separating real and imaginary parts gives

$$\varphi = x + \frac{\cos(\theta)}{r} = x + \frac{x}{x^2 + y^2} \tag{18.18}$$

$$\psi = y - \frac{\sin(\theta)}{r} = y - \frac{y}{x^2 + y^2} \tag{18.19}$$

Eqs.(18.18) and (18.19) are plotted in Fig. 18.2, where the streamline pattern for the doublet is clearly visible within the cylinder surface $\psi = 0$.

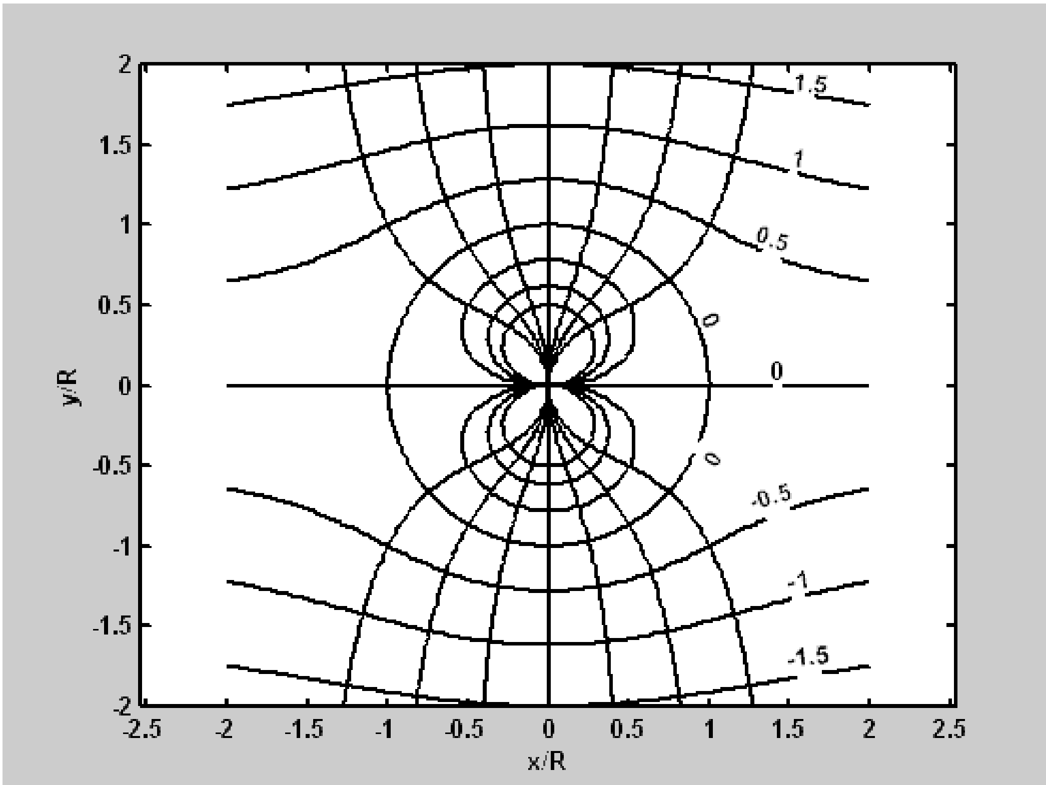
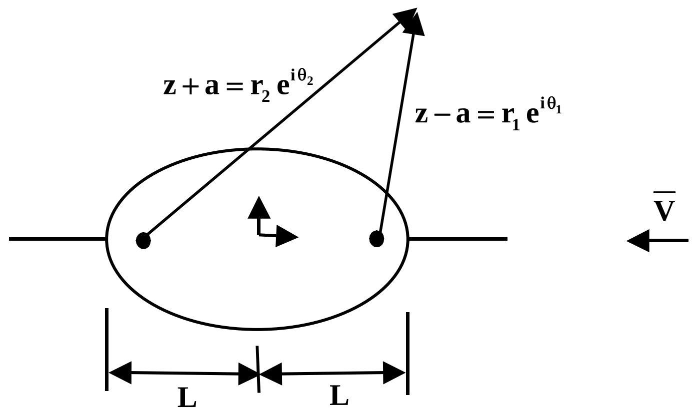


Fig. 18.2. Streamlines and potential lines for flow past a circular cylinder.



Flow past a Rankine body, shown in the above sketch, can be generated by placing a source and sink, each of equal strength, beside each other in uniform flow. If the source is at z = a + i0 and the sink is at z = -a + i0, and if the uniform flow is in the negative x direction, then the complex potential is given by

$$w = -Vz + \frac{q}{2\pi}\ln(z-a) - \frac{q}{2\pi}\ln(z+a) \tag{18.20}$$

and the complex conjugate of the velocity vector is

$$\frac{dw}{dz} \equiv u - i\,v = -V + \frac{q}{2\pi}\frac{1}{(z-a)} - \frac{q}{2\pi}\frac{1}{(z+a)} \tag{18.21}$$

Inserting the condition u = v = 0 at z = L + i0 in Eq.(18.21) gives an expression for q.

$$\frac{q}{2\pi} = \frac{Va}{2}\left[\left(\frac{L}{a}\right)^2 - 1\right] \tag{18.22}$$

Inserting Eq.(18.22) into Eq.(18.20) gives

$$\frac{w}{Va} = -\frac{z}{a} + \frac{1}{2}\left[\left(\frac{L}{a}\right)^2 - 1\right]\ln\left(\frac{z-a}{z+a}\right) \tag{18.23}$$

This suggests use of the following dimensionless variables:

$$(w^*, z^*, L^*) = \left(\frac{w}{Va}, \frac{z}{a}, \frac{L}{a}\right) \tag{18.24}$$

Rewriting Eq.(18.23) in these dimensionless variables gives

$$w = -z + \frac{\left(L^2 - 1\right)}{2} \ln\left(\frac{z-1}{z+1}\right) \tag{18.25}$$

Separating real and imaginary parts in Eq.(18.25) gives the following expressions for φ and ψ:

$$\varphi = -x + \frac{\left(L^2 - 1\right)}{2} \ln\left(\frac{r_1}{r_2}\right) = -x + \frac{\left(L^2 - 1\right)}{2} \ln\sqrt{\frac{(x-1)^2 + y^2}{(x+1)^2 + y^2}} \tag{18.26}$$

$$\psi = -y + \frac{\left(L^2 - 1\right)}{2}\left(\theta_1 - \theta_2\right) = -y + \frac{\left(L^2 - 1\right)}{2}\left[\tan^{-1}\left(\frac{y}{x-1}\right) - \tan^{-1}\left(\frac{y}{x+1}\right)\right] \tag{18.27}$$

Contours of constant φ and ψ are plotted in Fig. 18.3 for L = 1.2, where the Rankine body surface is seen to coincide with the oval shaped streamline $\psi = 0$. The heavy line within the body is a branch cut for the logarithmic terms in Eq.(18.20).

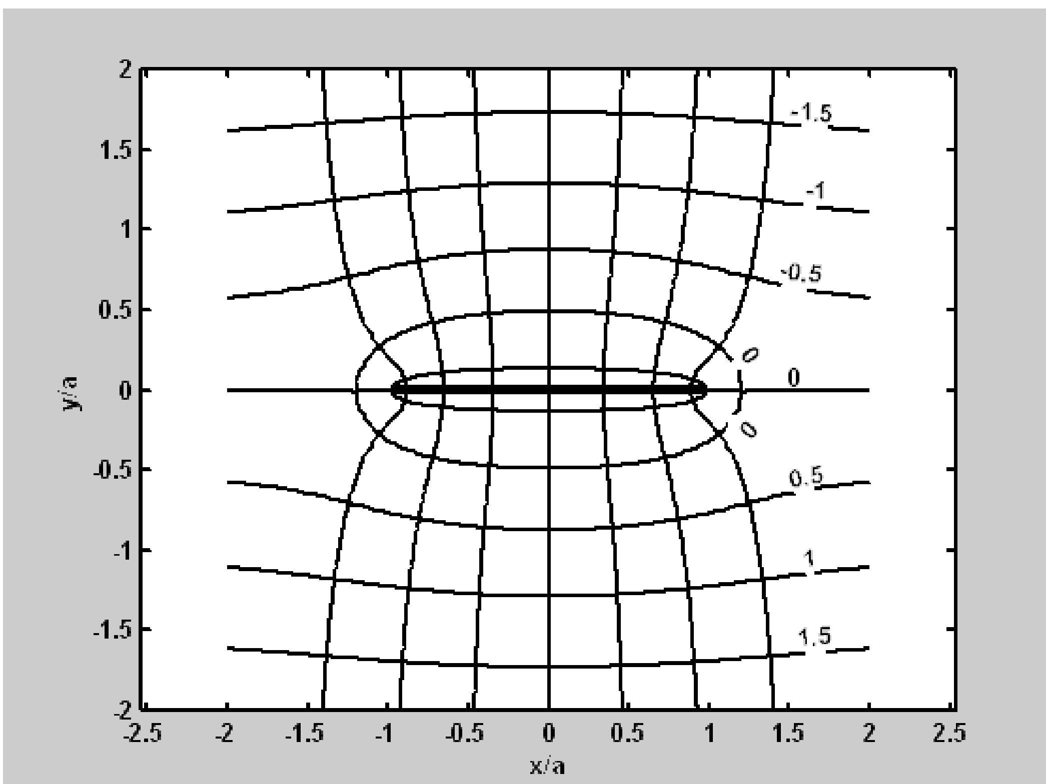


Fig. 18.3. Streamlines and potential lines for flow past a Rankine body.

This series of 18 lectures on the solution of second-order partial differential equations has provided a sound but modest introduction to the subject. Much more material could have been included in this introduction if time had permitted, and this is particularly true in the coverage of complex variable theory and its uses. For example, material covered in lecture 15 on analytic functions of a complex variable provides a foundation for the study of contour integration in the complex plane, which is a topic that is essential for learning how to compute both exact and asymptotic values for many definite integrals and for learning how to invert both Laplace and Fourier transforms when solving either ordinary or partial differential equations. After learning something about contour integration, you would be in a position to cover material on conformal mapping, a topic which is very useful for solving many problems in irrotational fluid motion, steady-state heat conduction and elasticity. Hildebrand (1976) gives a general introduction to these topics, and Carrier, Krook and Pearson (1966) cover the subject at a more advanced level. Milne-Thomson (1968) covers advanced applications of complex variables in irrotational flow problems, Polubarinova-Kochina (1962) gives extensive coverage of complex variable methods in groundwater flow problems, Carslaw and Jaeger (1959) give an introduction to the use of complex variables for steady-state head conduction problems and Muskhelishvili (1953) gives an extensive coverage of complex variable applications in elasticity.

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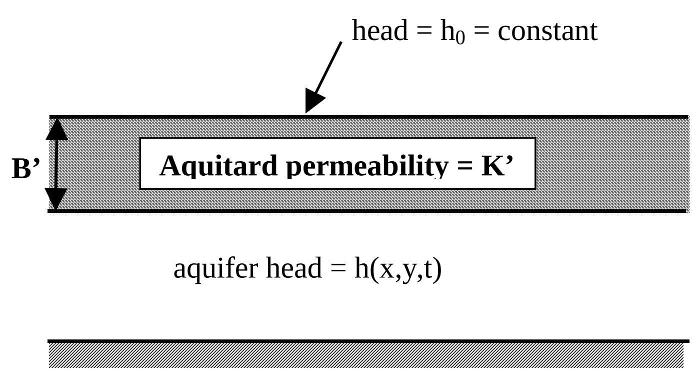
ENCI 302 (ENGINEERING MATHEMATICS)

PARTIAL DIFFERENTIAL EQUATIONS HOMEWORK ASSIGNMENTS

Lecture	***Problems***
1	**1**
2	**2**
3	**3**
4	**4**
5	**5, 6**
6	**7**
7	**8**
8	**9**
9	**10**
10	**11**
11	**12**
12	**13**
13	**14**
14	**15**
15	**16**
16	**17**
17	**None**
18	**18**

ENCI 302 (ENGINEERING MATHEMATICS)

PARTIAL DIFFERENTIAL EQUATIONS HOMEWORK PROBLEMS



1. Sometimes aquifer recharge, R, exists even when no free surface is present. This occurs when either the top or bottom aquifer boundary is semi-permeable, and the aquifer is referred to as a "leaky aquifer." (A semi-permeable formation is called an aquitard, and an impermeable formation is called an aquiclude.) Then, since the aquitard permeability, K', is much smaller than the aquifer permeability, K, flow in

the aquifer is horizontal but flow in the aquitard is vertical. Under these circumstances, a linear distribution of piezometric head exists across the aquitard in the vertical direction and Darcy's law gives $R = K'\left(h_0 - h\right)/B'$ where h = piezometric head in the aquifer, $h_0 =$ constant head on the far side of the aquitard and K' and B' are the permeability and thickness, respectively, of the aquitard. Inserting this result in Eqs.(1.10) and (1.12) gives the governing equations for a leaky aquifer:

$$T\nabla^2 h = S\frac{\partial h}{\partial t} + \left(\frac{K'}{B'}\right)\left(h - h_0\right)$$

$$T\nabla^2 s = S\frac{\partial s}{\partial t} + \left(\frac{K'}{B'}\right)s$$

(a) Suppose the a steady-flow problem is described by the following set of equations:

$$T\frac{\partial^2 s}{\partial x^2} = \left(\frac{K'}{B'}\right)s \qquad \left[s = s\left(x\right),\ 0 < x < L\right]$$

$$s\left(0\right) = s_0 \qquad \left(s_0 = \text{constant}\right)$$

$$\frac{\partial s\left(L\right)}{\partial x} = 0$$

Make a sketch of the physical problem described by these equations.

(b) Obtain the solution for s(x).

2. Consider the problem shown in Fig. 2.1, but replace the impermeable boundary at x = L with a reservoir that has the same constant free surface elevation as the river free surface at $x = 0$. Also assume that the impermeable aquifer boundary at z = 0 is replaced with an aquitard, so that the aquifer is leaky.

   (a) Write down the set of equations that gives a complete description of this problem. Remember to show ranges for the independent variables after each equation.

   (b) Use the variables given in Eq.(2.13) to rewrite the entire set of governing equations in dimensionless variables. The leakage term in the partial differential equation will lead to one additional dimensionless variable that is not shown in Eq.(2.13).

3. Use the orthogonality condition given by Eq.(3.8) to calculate Fourier coefficients a_n in the following Fourier series representations:[15]

(a) $2 = \sum_{n=1}^{\infty} a_n \sin\left(\alpha_n \frac{x}{L}\right) \qquad \left(0 < x < L,\ \alpha_n = n\pi\right)$

(b) $x = \sum_{n=1}^{\infty} a_n \sin\left(\alpha_n \frac{x}{L}\right) \qquad \left(0 < x < L,\ \alpha_n = n\pi\right)$

(c) Since $\sin\left(\alpha_n x / L\right)$ has a period of 2L and is an odd function of x, it is easy to sketch the function obtained from the right sides of these two Fourier series representations for $-\infty < x < \infty$. Make these sketches, and note the discontinuities that occur at x = 0 and x = L for part (a) and at x = L for part (b). Then read about the Gibbs phenomenon on pages 740-741 of Zill and Cullen to see the extremely large number of terms required for convergence and the resulting spikes that occur at discontinuities when a Fourier series is used to represent a discontinuous function. **We will avoid this problem when solving partial differential equations with a Fourier series by ensuring that the dependent variable satisfies homogeneous boundary conditions.** (This will be sufficient to ensure that the periodic function obtained from the Fourier series is continuous for $-\infty < x < \infty$**.)**

4. Use a half-range Fourier series to solve the problem that you formulated in dimensionless variables for question 2 (b).

---

[15] You will need to use an integration by parts to evaluate an integral needed for part (b). Also, note that $\sin\left(\alpha_n x / L\right) = 0$ at both x = 0 and x = L, which means that conditions required for validity of the orthogonality conditions are satisfied.

5. Write a Visual Basic program in a module that calculates numerical values from the solution that you obtained in question 4. Then write a second program that calculates numerical values for a pulse that extends from t = 0 to t = Δ.
6. Use the programs in question 5 to make plots corresponding to Figs. 5.1 and 5.2. In your plots set $K'/B' = 0.005$ days^{-1} and let all remaining variables equal the dimensional values used in Figs. 5.1 and 5.2.
7. Obtain the plot corresponding to the plot in Fig. 6.1 for the pulse solution considered in questions 5 and 6.
8. The solution that you obtained in question 4 for a leaky aquifer cannot be used to find the solution for $L \to \infty$, apparently because the reservoir at x = L cannot be placed at infinity. (The drawdown magnitude should increase monotonically as x increases from zero to infinity, but the plot you made in question 6 has the drawdown magnitude increasing to a relative maximum at x = L/2 before decreasing to zero at x = L.) However, the solution for $L \to \infty$ can be found by replacing the reservoir at x = L with an impermeable clay embankment. The resulting problem description follows:

$$\left(s^*, x^*, t^*, K^*\right) = \left(\frac{sT}{RL^2}, \frac{x}{L}, \frac{tT}{SL^2}, \frac{\left(K'/B'\right)L^2}{T}\right)$$

$$\frac{\partial^2 s}{\partial x^2} = \frac{\partial s}{\partial t} + K\,s + 1 \qquad \left(0 < x < 1,\ 0 < t < \infty\right)$$

$$s\left(0, t\right) = 0 \qquad \left(0 < t < \infty\right)$$

$$\frac{\partial s\left(1, t\right)}{\partial x} = 0 \qquad \left(0 < t < \infty\right)$$

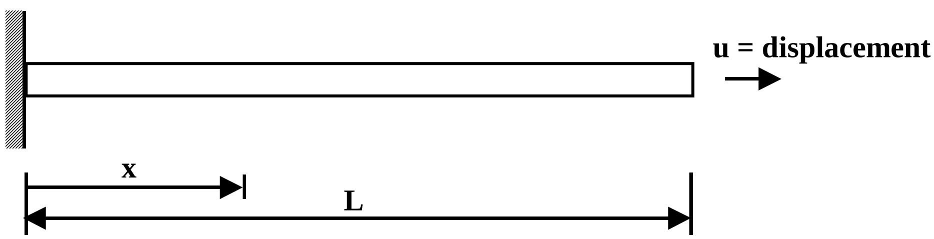
$$s\left(x, 0\right) = 0 \qquad \left(0 < x < 1\right)$$

Solve this problem in dimensionless variables, replace the dimensionless variables in your solution with their dimensional form and then take the limit $L \to \infty$ to obtain a dimensional solution corresponding to Eq.(7.8). Finally, make substitutions in the integral to obtain a dimensionless solution corresponding to Eq.(7.10).

9. Use Simpson's rule to evaluate the solution in question 8 and obtain a plot corresponding to the one shown in Fig. 7.2.

10. Solve the problem given by Eqs.(9.1)-(9.4) if s(L,t) = 0 replaces the boundary condition given by Eq.(9.3).

11. Solve the problem given by Eqs.(10.1)-(10.4) if $\partial s(L,t)/\partial x = 0$ replaces the boundary condition given by Eq.(10.3). Then use your solution for finite L to obtain the solution for $L \to \infty$.



12. If a beam or bar is subjected to an axial force, its longitudinal displacement, u(x,t), is a solution of the following equation:

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad \left(c = \sqrt{E/\rho},\ 0 < x < L,\ 0 < t < \infty\right)$$

where E = modulus of elasticity and $\rho =$ mass density of the beam or bar. If the bar end at x = 0 is fixed, if other end at x = L is given a small displacement, u_0, and if motion starts from rest so that both its longitudinal displacement, u, and its longitudinal velocity, $\partial u/\partial t$, are zero at t = 0, then the above partial differential equation must be solved simultaneously with the following boundary and initial conditions:

$$
\begin{aligned}
&u(0,t)=0 && (0<t<\infty)\\
&u(L,t)=u_0 && (0<t<\infty,\ u_0=\text{constant})\\
&u(x,0)=0 && (0<x<L)\\
&\frac{\partial u(x,0)}{\partial t}=0 && (0<x<L)
\end{aligned}
$$

Rewrite these equations with an appropriately chosen set of dimensionless variables, use Eq.(10.10) to reformulate the problem with a new dependent variable, $\varphi(x,t)$, that has homogeneous boundary conditions and use a half-range Fourier series to solve for $\varphi(x,t)$. Finally, write a user-defined function in Visual Basic to calculate u and use it to plot dimensionless values of u versus t for a dimensionless value of x equal to ½.

13. Use the trigonometric identity $\sin(A-B)+\sin(A+B)=2\sin(A)\cos(B)$ to show that the solution of question 12 can be rewritten in the alternative form $u(x,t)=f(x-ct)+g(x+ct)$ where c = 1 (since the dimensionless set of equations that you solved had c = 1).

14. Eq.(11.17) shows that free surface fluctuations created by the tide or by an approaching train of tsunami waves in a long straight harbour can be modelled by solving the following "steady-state" problem:

$$
\begin{aligned}
&c^2\frac{\partial^2 h}{\partial x^2}=\frac{\partial^2 h}{\partial t^2} && (0<x<L,\ -\infty<t<\infty)\\
&h(0,t)=a\ \cos(\omega t) && (-\infty<t<\infty)\\
&\frac{\partial h(L,t)}{\partial x}=0 && (-\infty<t<\infty)
\end{aligned}
$$

where the last equation is determined by inserting $u'(L,t)=0$ for $-\infty<t<\infty$ in Eq.(11.13). Rewrite these equations in an appropriately chosen set of dimensionless variables, calculate the steady-state solution and obtain an expression that gives the harbour lengths required to create resonance. Note that the governing equations, which were derived by assuming relatively small values for velocities and free surface displacements, become invalid when h(x,t) becomes large. Therefore, velocities and free surface displacements can become relatively large, but never infinite, when resonant conditions are approached.

15. Calculate all roots of the following equations:

$$z^2 - a^2 = 0$$
$$z^4 - a^4 = 0$$

where a is a positive real number. How many different roots do you suspect exist for the equation $z^n - a^n = 0$?

16. Consider the following two functions:

(a) $f(z) = z^{1/3}$

(b) $f(z) = \ln(z)$

Rewrite the expressions for $f(z)$ in polar coordinates by setting $z = r e^{i\theta}$. Then surround the branch point at the origin with a small circle, run a branch cut along the negative real axis from the branch point at the origin to the second branch point at $z = \infty$ and choose the branch defined by $-\pi \le \theta \le \pi$. Calculate the discontinuity in $f(z)$ along the branch cut for both (a) and (b).

17. Consider the complex velocity potential $w(z) = z^2$. Set $w = \varphi + i\psi$ and $z = x + i y$ and separate real and imaginary parts to obtain $\varphi = x^2 - y^2$ and $\psi = 2xy$. Since $\psi = 0$ on both the positive real and positive imaginary axes, this is the potential for flow in the corner contained in the first quadrant of the z plane.

(a) Calculate dw/dz and use the result to show that the velocity components are given by u = 2x and v = -2y. Thus, along the positive real axis we have u = 2x and v = 0, while u = 0 and v = -2y along the positive imaginary axis.

Furthermore, u = v = 0 at x = y = 0, so the origin is a point of stagnation where dw/dz = 0. This means that curves of constant φ and constant ψ may not be orthogonal at the origin.

(b) Make a spreadsheet plot of the curves $\varphi = (-2, -1, 0, 1, 2)$ and $\psi = (1, 2, 3)$ for 0<x<2 and 0<y<2. After you plot these curves, adjust the axes scales so that 0<x<4 and 0<y<2. This should give you a **printed** plot that is undistorted, and you can observe that curves of constant φ and constant ψ are orthogonal everywhere except at the origin. Hint: Carry out the calculations for x and y by putting the expressions for φ and ψ in the forms $y = \sqrt{x^2 - \varphi}$ and $y = \psi / (2x)$.

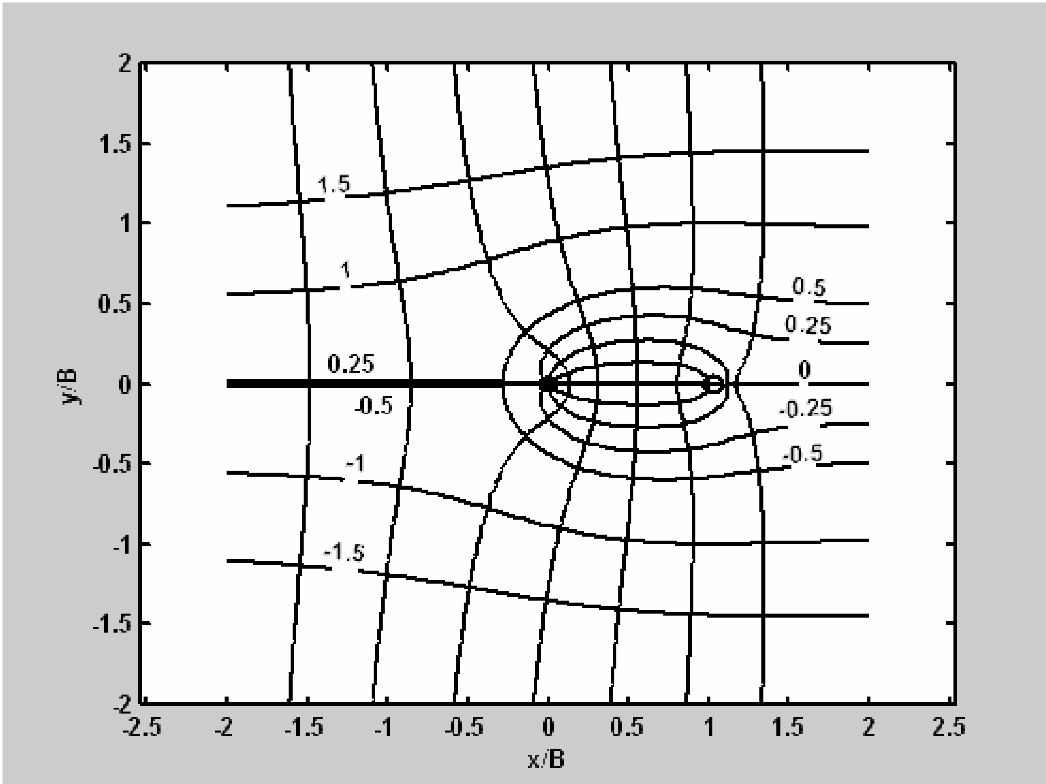
18. The boundary streamline for any body generated by placing sources and sinks in a uniform flow can close upon itself at a finite distance from the origin only if the total flow emitted by all sources and the total flow absorbed by all sinks within the body interior are identical. Write down the expression for the complex potential obtained by superimposing a source of strength q at the origin, a sink of strength q/2 at $z = B + 0i$ and a uniform flow in the positive x direction. Since a net flow of q/2 within the body must escape to infinity, and since velocities at infinity (both within and outside the body) are given by the constant uniform flow velocity, V, the asymptotic body width at infinity, B, is obtained from the equation $q/2 = VB$. Use this expression for q to rewrite the complex potential in terms of the following dimensionless variables:

$$(w^*, z^*) = \left(\frac{w}{VB}, \frac{z}{B} \right)$$

Then use the following Matlab program in an M-file to plot streamlines and potential lines for the dimensionless potential:

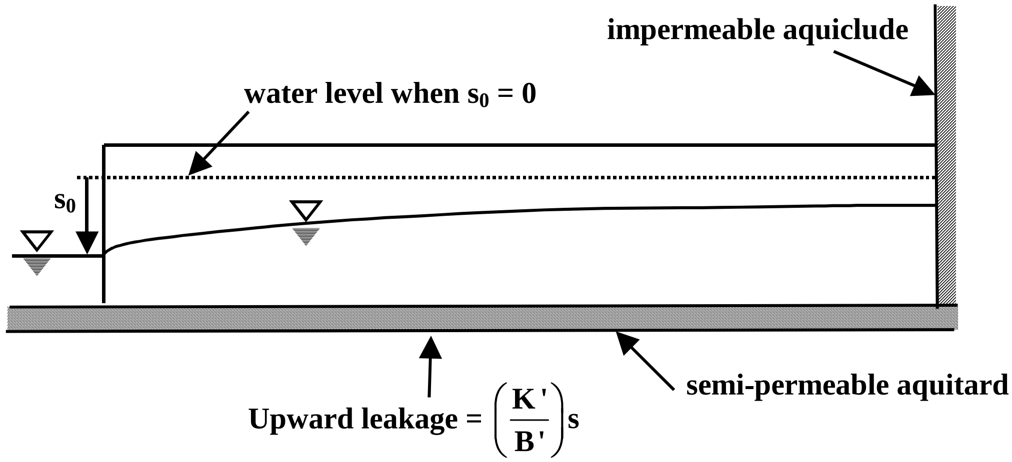
```
%This plots streamlines and potential lines for an irrotational flow
problem.
function plot_w
X=linspace(-2,2,120); Y=linspace(-2,2,120); [x,y]=meshgrid(X,Y);
phi=x+log(sqrt(x.^2+y.^2))/pi-log(sqrt((x-1).^2+y.^2))/(2*pi);
[c,h]=contour(x,y,phi,[-1.5,-1,-0.5,0,0.5,1,1.5],'k');%Gives handles c and
%h to contours
set(h,'linewidth',2); %Uses handle h to change line width.
hold on;
psi=y+atan2(y,x)/pi-atan2(y,x-1)/(2*pi);
[c,h]=contour(x,y,psi,[-1.5,-1,-0.5,-0.25,0,0.25,0.5,1,1.5],'k');%Gives
%handles c and h to contours
set(h,'linewidth',2); %Uses handle h to change line width.
clabel(c,h,'manual','fontweight','bold');
axis('equal'); xlabel('\bfx/B'); ylabel('\bfy/B');
set(gca,'linewidth',2,'fontweight','bold');%gca returns axes handles. This
%sets axes line width and axes label font weight.
```

Call the program by typing plot_w in the Matlab command window. The ‘manual’ command in the clabel statement allows you to print a contour label on any streamline simply by centering the cross hairs on the streamline and left clicking. Double clicking terminates this labeling process. The resulting plot is shown below.



ANSWERS FOR HOMEWORK PROBLEMS

1. (a)



(b) $s(x) = s_0 \dfrac{e^{\alpha(L-x)} + e^{-\alpha(L-x)}}{e^{\alpha L} + e^{-\alpha L}} \equiv s_0 \dfrac{\cosh\left[\alpha(L-x)\right]}{\cosh(\alpha L)}$ $\qquad \left(\alpha \equiv \sqrt{\dfrac{K'/B'}{T}}\right)$

2. (a)

$$T\frac{\partial^2 s}{\partial x^2} = S\frac{\partial s}{\partial t} + \left(\frac{K'}{B'}\right)s + R \qquad (0 < x < L,\ 0 < t < \infty)$$
$$s(0,t) = 0 \qquad (0 < t < \infty)$$
$$s(L,t) = 0 \qquad (0 < t < \infty)$$
$$s(x,0) = 0 \qquad (0 < x < L)$$

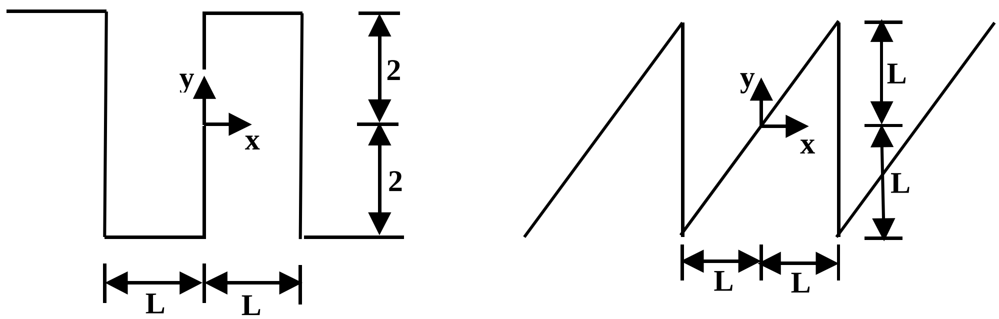
(b)

$$(s^*, x^*, t^*, K^*) = \left(\frac{sT}{RL^2}, \frac{x}{L}, \frac{tT}{SL^2}, \frac{(K'/B')L^2}{T}\right)$$
$$\frac{\partial^2 s}{\partial x^2} = \frac{\partial s}{\partial t} + K\,s + 1 \qquad (0 < x < 1,\ 0 < t < \infty)$$
$$s(0,t) = 0 \qquad (0 < t < \infty)$$
$$s(1,t) = 0 \qquad (0 < t < \infty)$$
$$s(x,0) = 0 \qquad (0 < x < 1)$$

3. (a) $a_n = 4\dfrac{1-(-1)^n}{\alpha_n}$

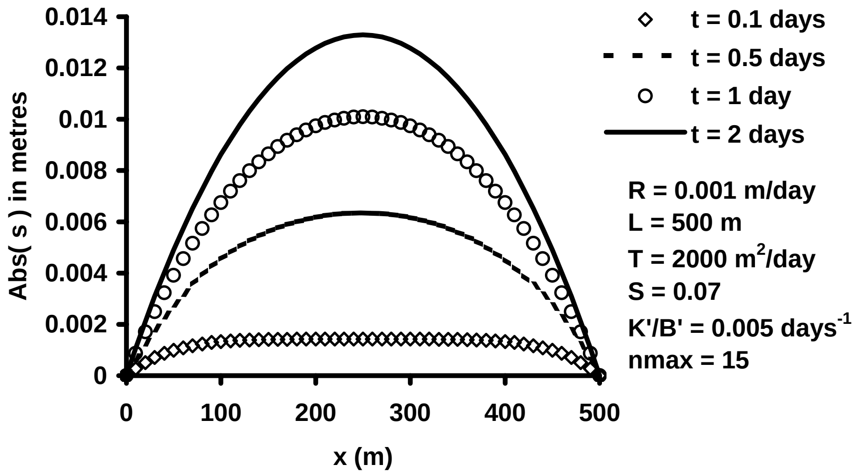
(b) $a_n = 2L\dfrac{(-1)^{n+1}}{\alpha_n}$

(c)



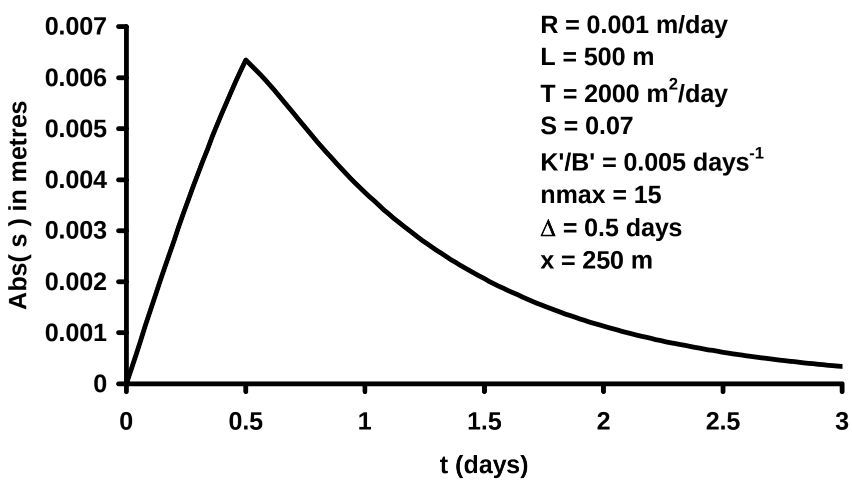
4. $s(x,t) = -2\sum_{n=1}^{\infty}\left[1-(-1)^n\right]\dfrac{\sin(\alpha_n x)}{\alpha_n\left(K+\alpha_n^{\;2}\right)}\left[1-e^{-\left(K+\alpha_n^{\;2}\right)t}\right]$

The negative value for s(x,t) indicates a rise in the elevation of the free surface.



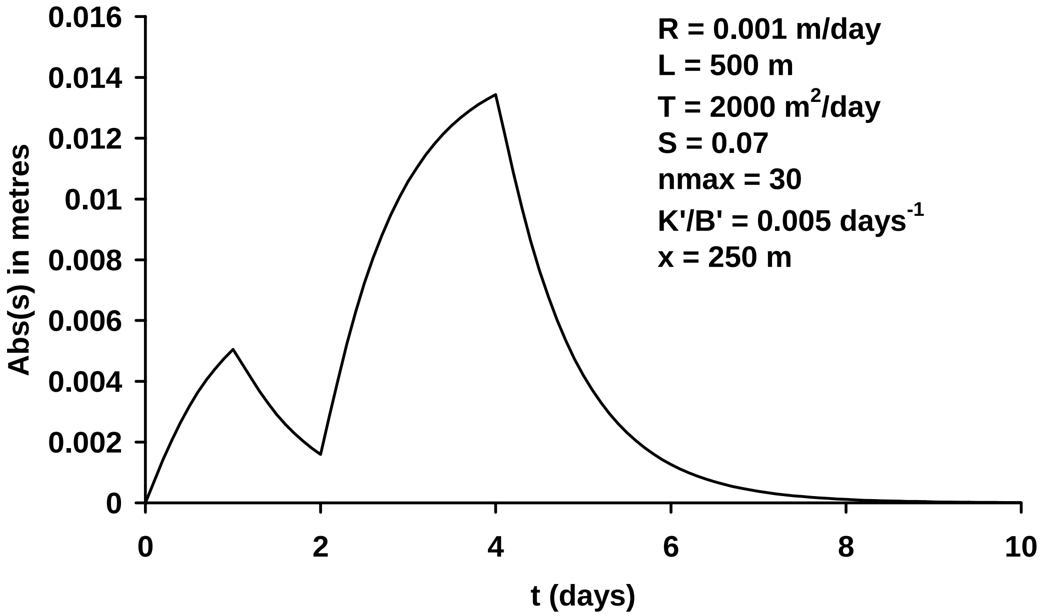
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
Abs(s) in metres
0
100
200
300
400
500
x (m)
t = 0.1 days
t = 0.5 days
t = 1 day
t = 2 days
R = 0.001 m/day
L = 500 m
T = 2000 m2/day
S = 0.07
K'/B' = 0.005 days-1
nmax = 15

5. and 6.



0.007
0.006
0.005
0.004
0.003
0.002
0.001
0
Abs(s) in metres
0
0.5
1
1.5
2
2.5
3
t (days)
R = 0.001 m/day
L = 500 m
T = 2000 m2/day
S = 0.07
K'/B' = 0.005 days-1
nmax = 15
Δ = 0.5 days
x = 250 m

7.

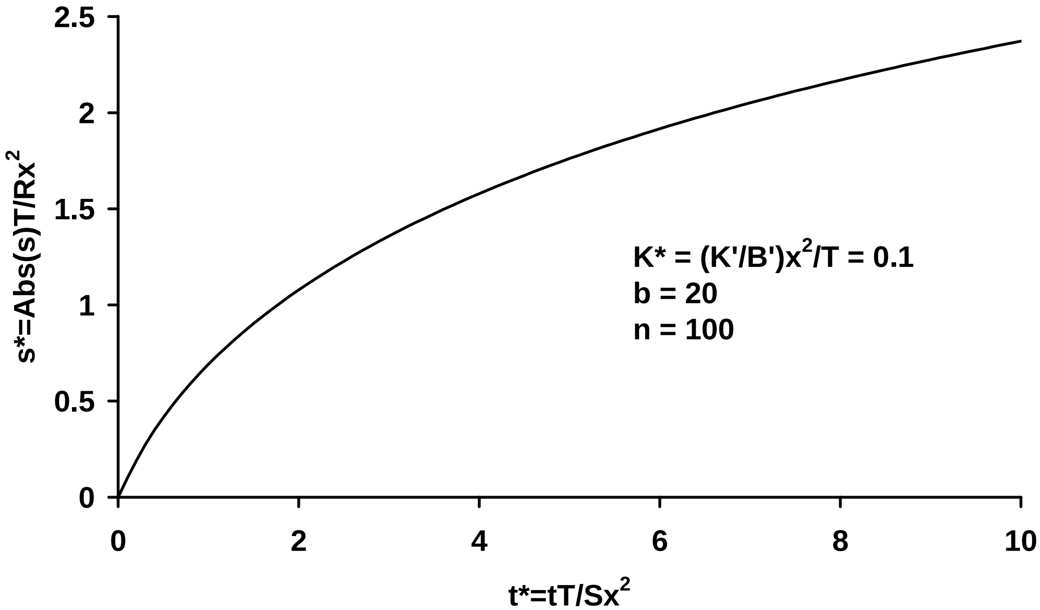


8. $$s = -\frac{2}{\pi}\frac{R}{T}\int_0^\infty \frac{\sin(\xi x)}{\xi\left(\frac{K'/B'}{T}+\xi^2\right)}\left[1-e^{-\left(\frac{K'/B'}{T}+\xi^2\right)\frac{tT}{S}}\right]d\xi$$

$$(s^*, t^*, K^*) = \left(\frac{sT}{Rx^2}, \frac{tT}{Sx^2}, \frac{(K'/B')x^2}{T}\right)$$

$$s = -\frac{2}{\pi}\int_0^\infty \frac{\sin(u)}{u(K+u^2)}\left[1-e^{-(K+u^2)t}\right]du$$

9. The numerical solution is plotted below for K* = 0.1.



10. $\left(s^*, x^*, t^*\right) = \left(-\frac{s}{H}, \frac{x}{L}, \frac{tT}{SL^2}\right)$

$$s\left(x,t\right) = 2\sum_{n=1}^{\infty}\left[1-\left(-1\right)^n\right]\frac{\sin\left(\alpha_n x\right)}{\alpha_n}e^{-t\alpha_n^2} \qquad \left(\alpha_n = n\pi\right)$$

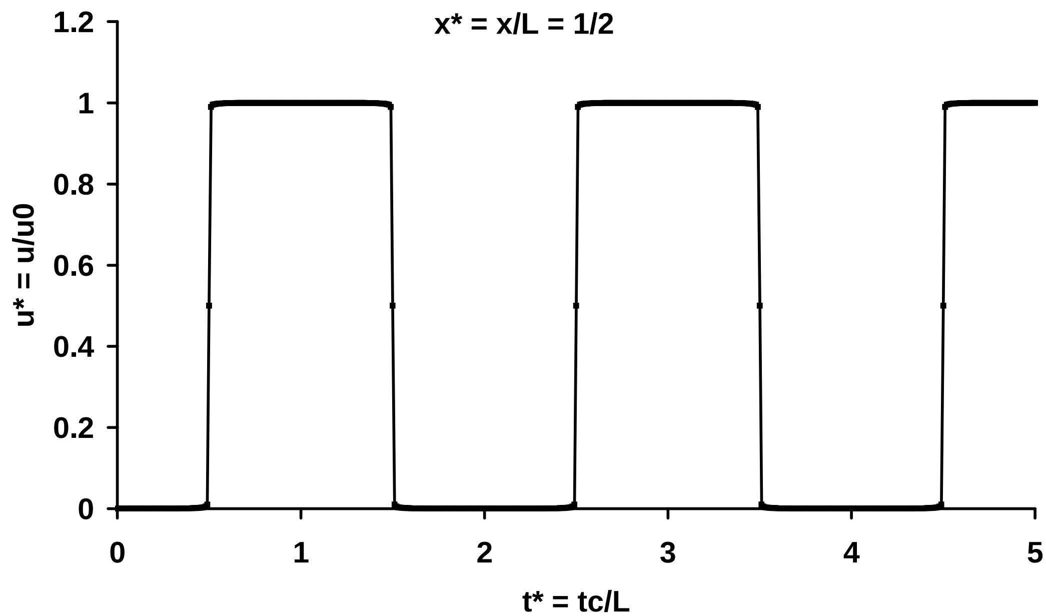
11.

$$\left(s^*, x^*, t^*\right) = \left(-\frac{s}{H}, \frac{x}{L}, \frac{tT}{SL^2}\right)$$

$$s\left(x,t\right) = 1 - 2\sum_{n=1}^{\infty}\frac{\sin\left(\alpha_n x\right)}{\alpha_n}e^{-t\alpha_n^2} \qquad \left[\alpha_n = \left(2n-1\right)\frac{\pi}{2}\right]$$

12. $u\left(x,t\right) = x + 2\sum_{n=1}^{\infty}\left(-1\right)^n\frac{\sin\left(\alpha_n x\right)}{\alpha_n}\cos\left(\alpha_n t\right) \quad \left(\alpha_n = n\pi\right)$

The following plot of this solution set x = ½ and used 1000 terms in the series. Notice the oscillations, which are typical of solutions of the wave equation.



13. $u(x,t)=\left\{\frac{1}{2}(x-t)+\sum_{n=1}^{\infty}(-1)^n\frac{\sin[\alpha_n(x-t)]}{\alpha_n}\right\}+\left\{\frac{1}{2}(x+t)+\sum_{n=1}^{\infty}(-1)^n\frac{\sin[\alpha_n(x+t)]}{\alpha_n}\right\}$

14. $h(x,t)=\frac{\cos[\omega(1-x)]}{\cos(\omega)}\cos(\omega t)$ where $(h^*,x^*,t^*,\omega^*)=\left(\frac{h}{a},\frac{x}{L},\frac{tc}{L},\frac{\omega L}{c}\right)$

Since wave amplification is given by $|h(x,t)/h(0,t)|=|\cos[\omega(1-x)]|/|\cos(\omega)|$, infinite wave amplification occurs when

$$\omega^*\equiv\frac{\omega L}{c}=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2},\cdots$$

where $c\equiv\sqrt{gh_0}$.

15. The roots of $z^2-a^2=0$ are $z_1=a$ and $z_2=-a$.

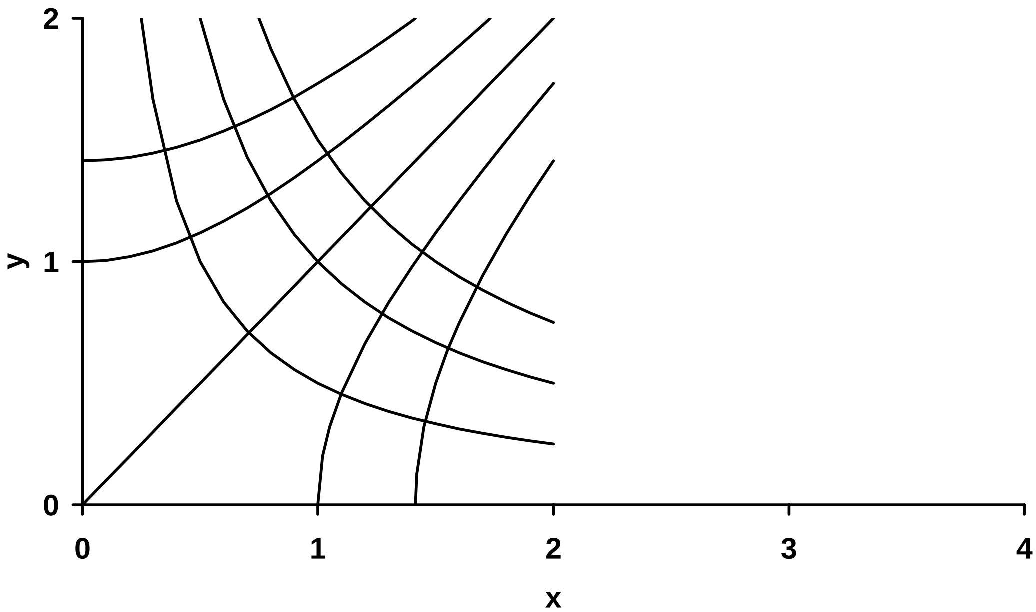
The roots of $z^4-a^4=0$ are $z_1=a$, $z_2=ia$, $z_3=-a$ and $z_4=-ia$.

The equation $z^n-a^n=0$ has n different roots.

16. (a) Discontinuity $= i\, r^{1/3}\sqrt{3}$.

(b) Discontinuity $= 2\pi i$.

17.



18.

$$\left(w^*, z^*\right) = \left(\frac{w}{VB}, \frac{z}{B}\right)$$
$$w^* = z^* + \frac{1}{\pi}\ln\left(z^*\right) - \frac{1}{2\pi}\ln\left(z^* - 1\right)$$
$$\varphi = x + \frac{1}{\pi}\ln\sqrt{x^2 + y^2} - \frac{1}{2\pi}\ln\sqrt{\left(x-1\right)^2 + y^2}$$
$$\psi = y + \frac{1}{\pi}\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2\pi}\tan^{-1}\left(\frac{y}{x-1}\right)$$

where an additive constant has been neglected. The Matlab program and plot are shown in the problem statement.